Independence

Today, start with a cool program


G_1
G_2
G_3
G_4
G_5
T

G_1 G_2 G_3 G_4 G_5 T

dna.txt — dna

dna.txt

```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
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False,True,False,False,True,True
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True,False,False,True,False,False
True,True,False,True,True,True
True,True,False,False,True,True
True,True,False,True,True,True
False,False,False,True,False,False
```
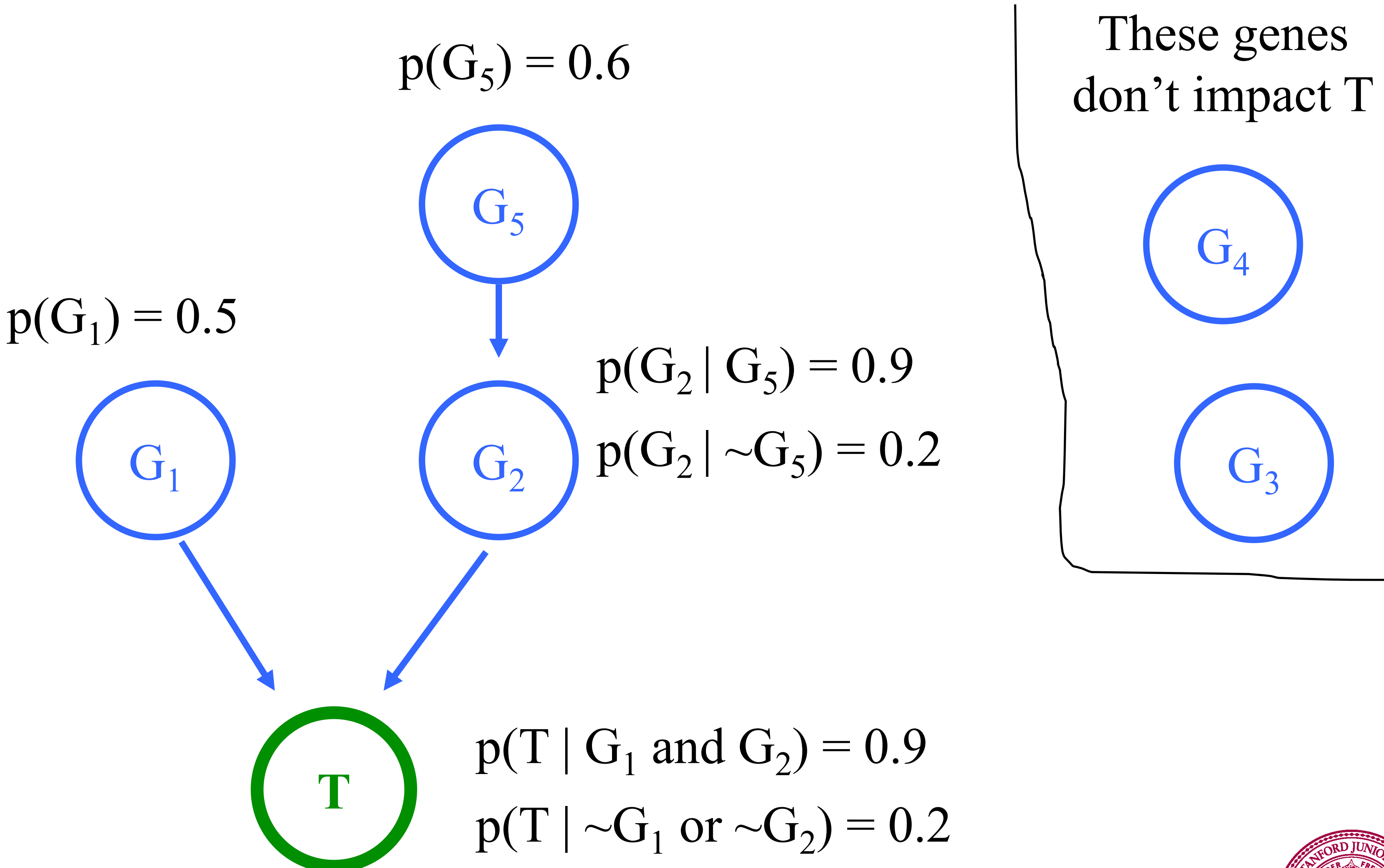

100,000 samples

6 observations per sample

Discovered Pattern



We’ve gotten ahead of ourselves

Source: The Ho

Start at the beggining

Source: The Ho

And vs Condition

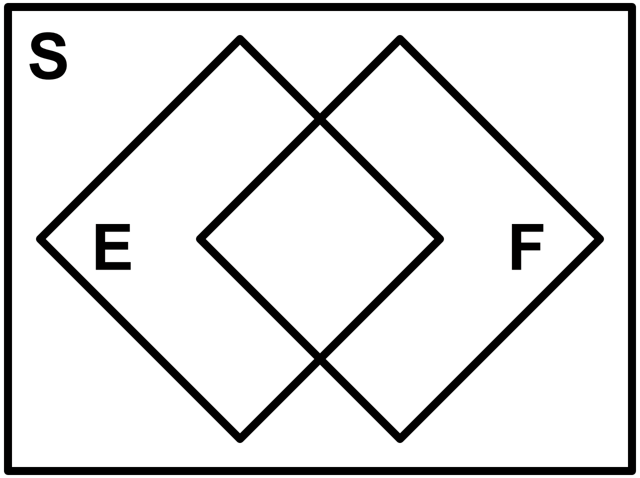
$$P(AB) \quad \text{vs} \quad P(A|B)$$

$$P(AB) = P(A|B)P(B)$$

Sets Review

Set Operations Review

- Say E and F are subsets of S

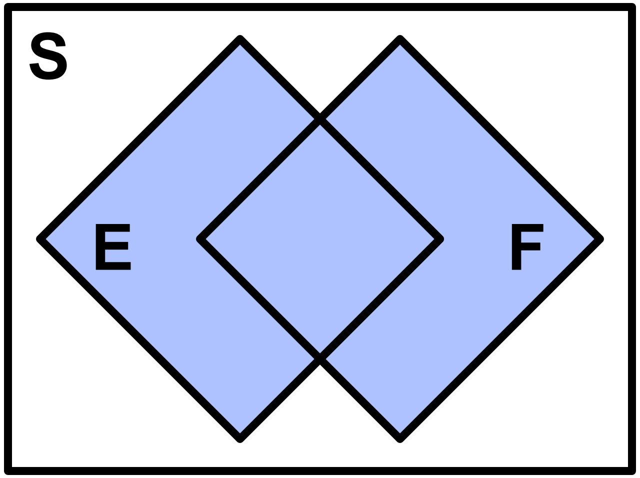


Set Operations Review

- Say E and F are events in S

Event that is in E <u>or</u> F

E ∪ F



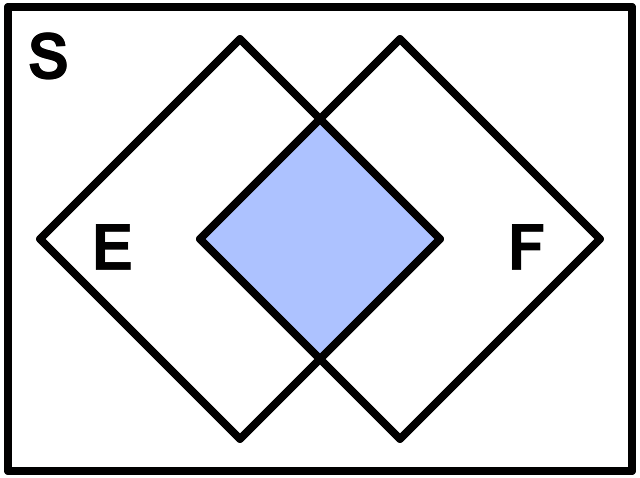
- $S = \{1, 2, 3, 4, 5, 6\}$ die roll outcome
- $E = \{1, 2\}$ $F = \{2, 3\}$ $E \cup F = \{1, 2, 3\}$

Set Operations Review

- Say E and F are events in S

Event that is in E <u>and</u> F

E ∩ F *or* **EF**



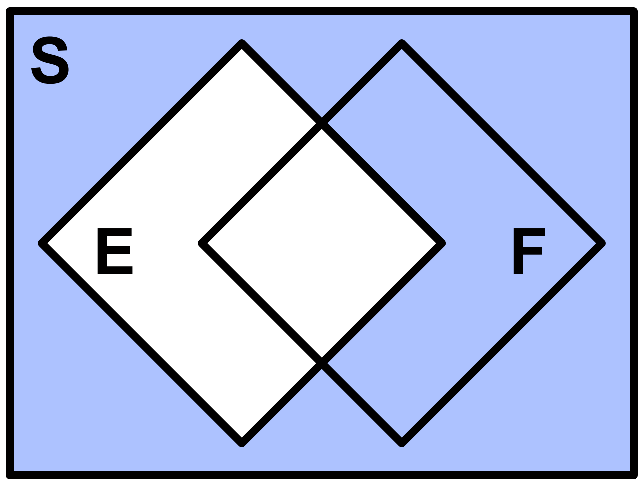
- S = {1, 2, 3, 4, 5, 6} die roll outcome
- E = {1, 2} F = {2, 3} E F = {2}
- **<u>Note:</u>** <u>*mutually exclusive*</u> events means E F = ∅

Set Operations Review

- Say E and F are events in S

  Event that is not in E (called complement of E)

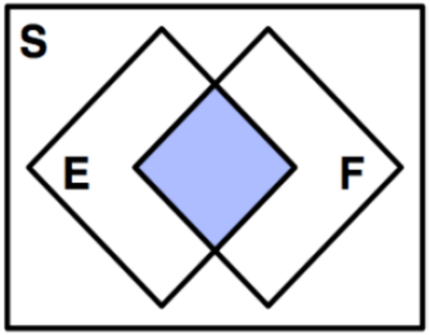
  E^c *or* **~E**



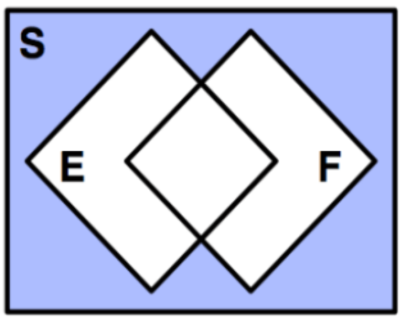
  - S = {1, 2, 3, 4, 5, 6}   die roll outcome
  - E = {1, 2} E^c = {3, 4, 5, 6}

Which is the correct picture for $\mathbf{E^c \cap F^c}$

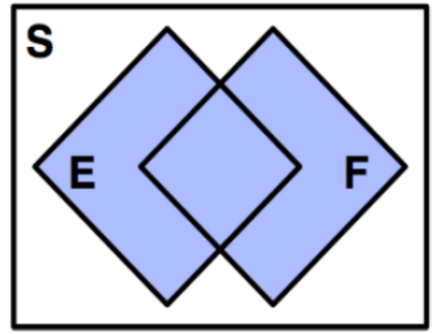
A



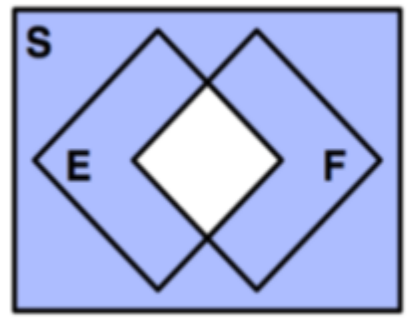
B



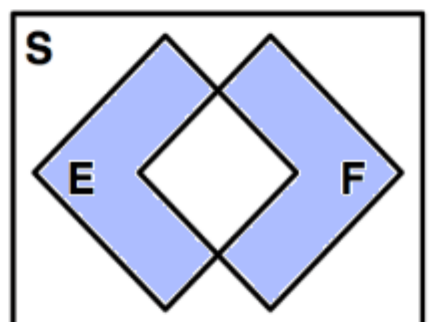
C



D



E

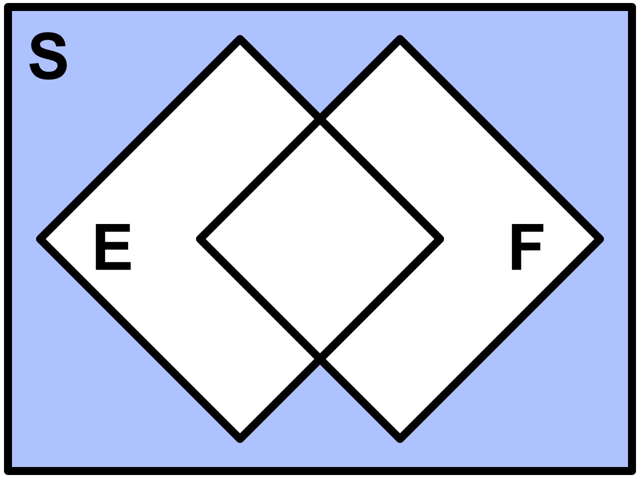


Set Operations Review

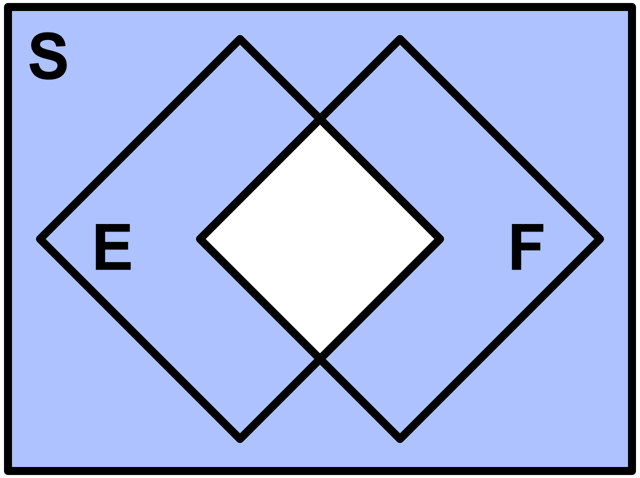
- Say E and F are events in S

DeMorgan's Laws

$(E \cup F)^c = E^c \cap F^c$

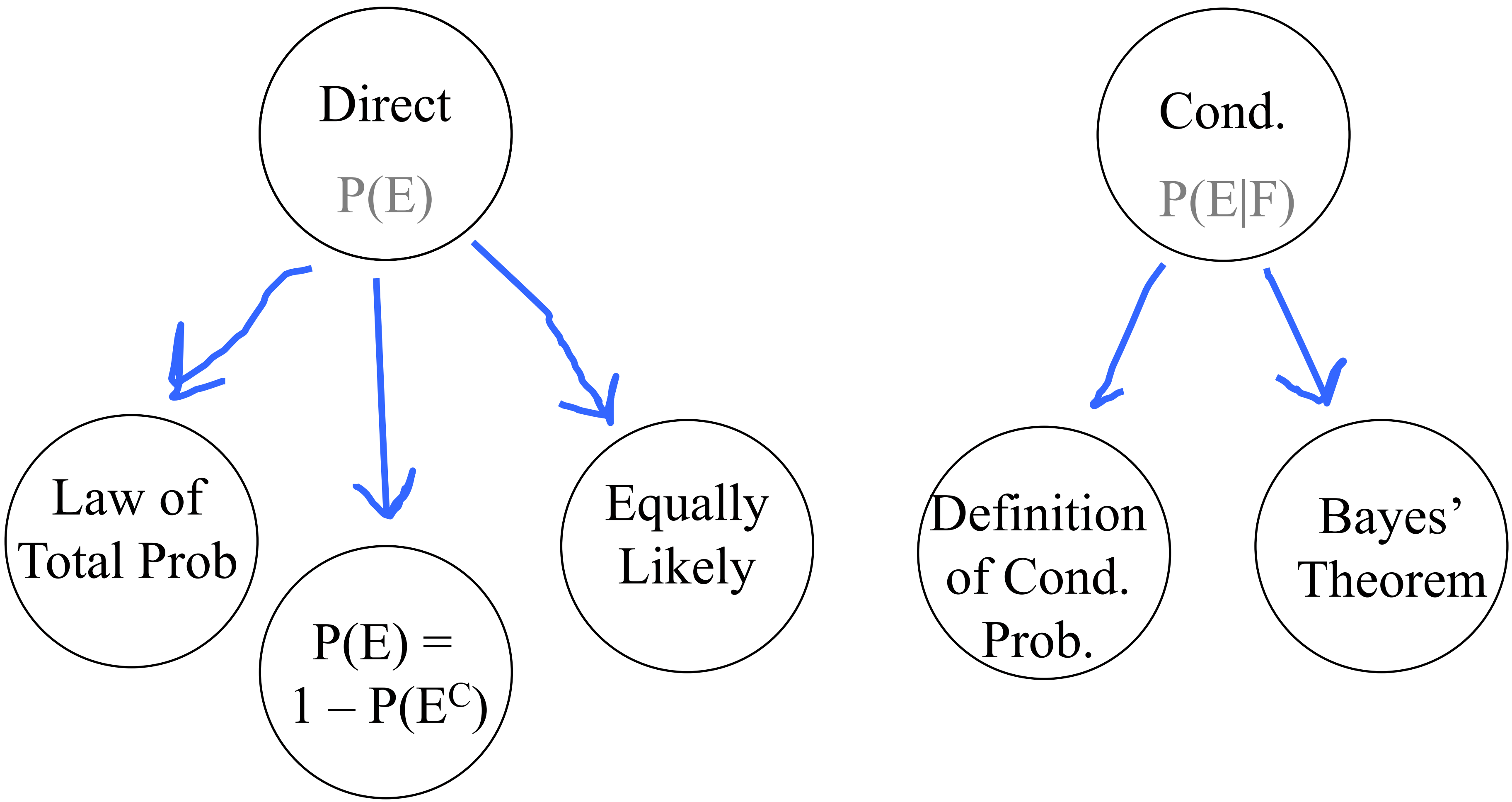


$(E \cap F)^c = E^c \cup F^c$



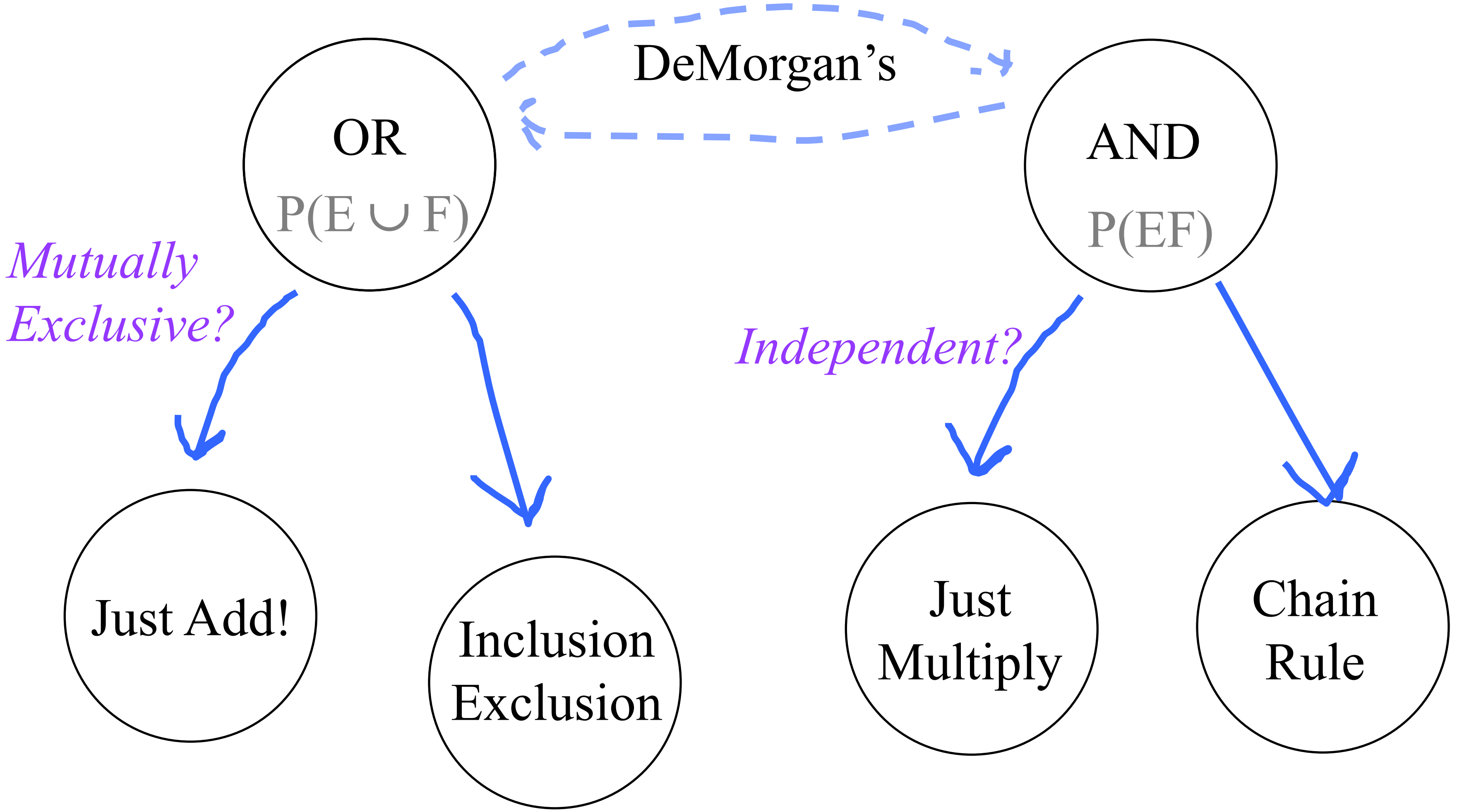
Core probability in two slides?

So Far



Direct
P(E)
Law of
Total Prob
P(E) =
1 – P(E^C)
Equally
Likely
Cond.
P(E|F)
Definition
of Cond.
Prob.
Bayes'
Theorem

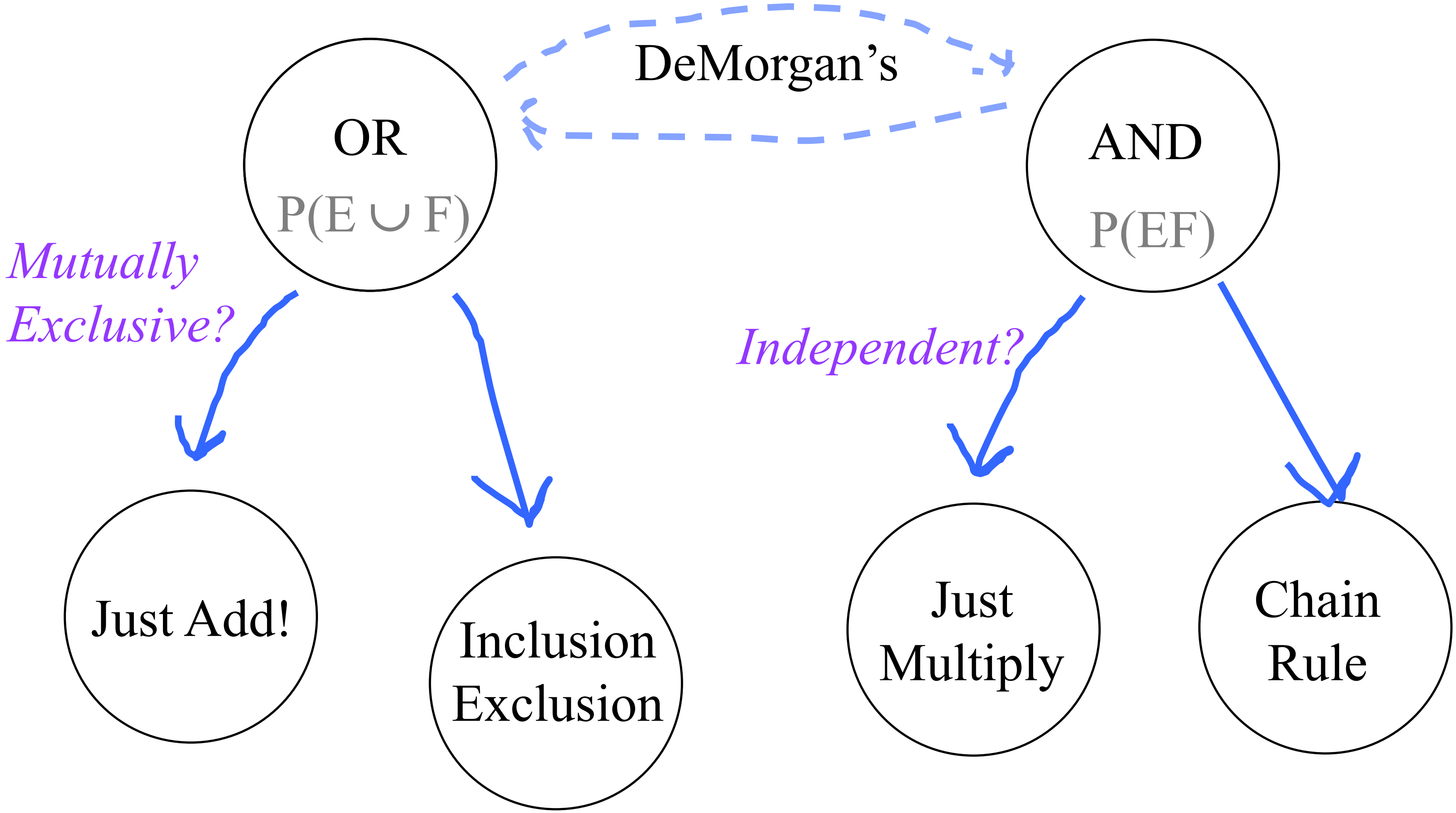
Today



DeMorgan's
OR
P(E ∪ F)
AND
P(EF)
Mutually Exclusive?
Independent?
Just Add!
Inclusion Exclusion
Just Multiply
Chain Rule

End Review

Today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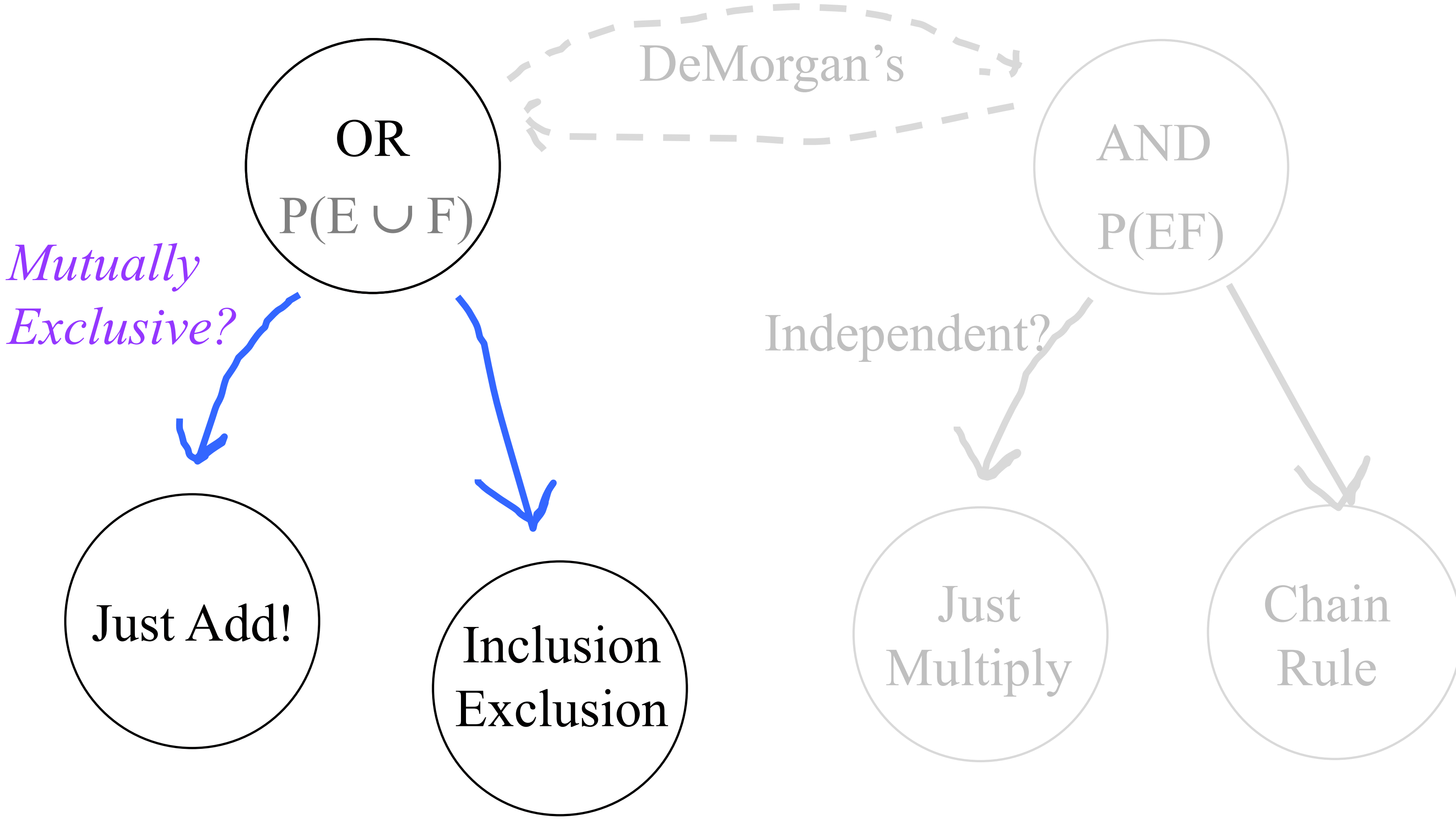
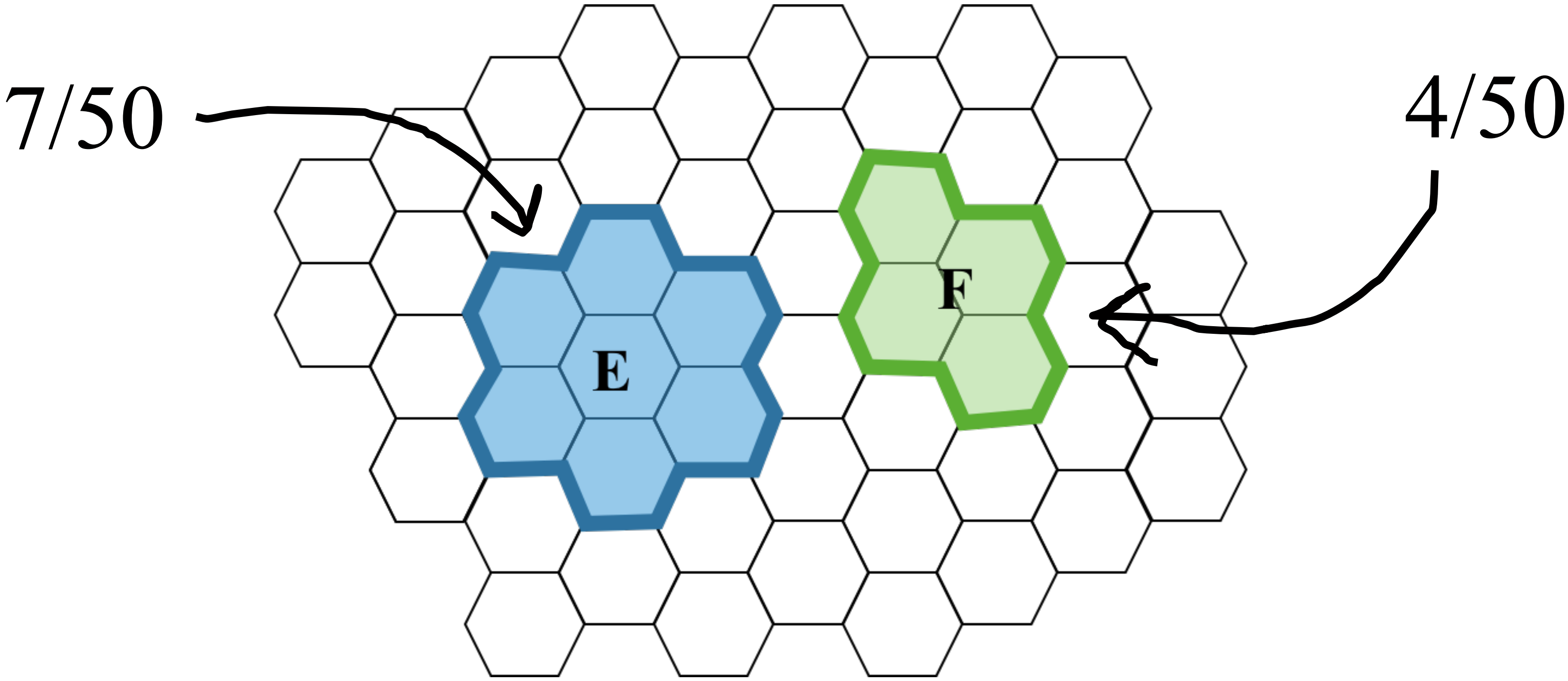
Today



OR
P(E ∪ F)
DeMorgan's
AND
P(EF)
Mutually
Exclusive?
Independent?
Just Add!
Inclusion
Exclusion
Just
Multiply
Chain
Rule

Probability of “OR”

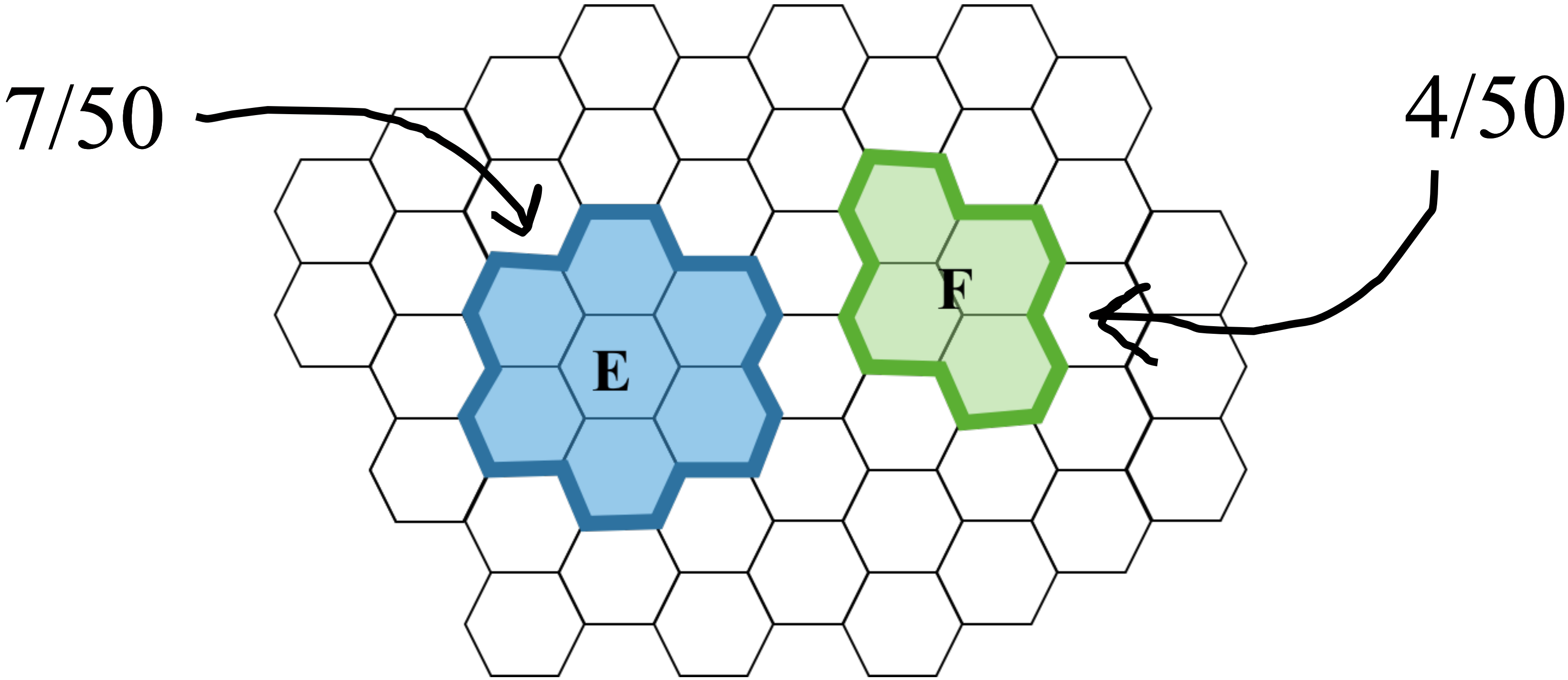
OR with Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

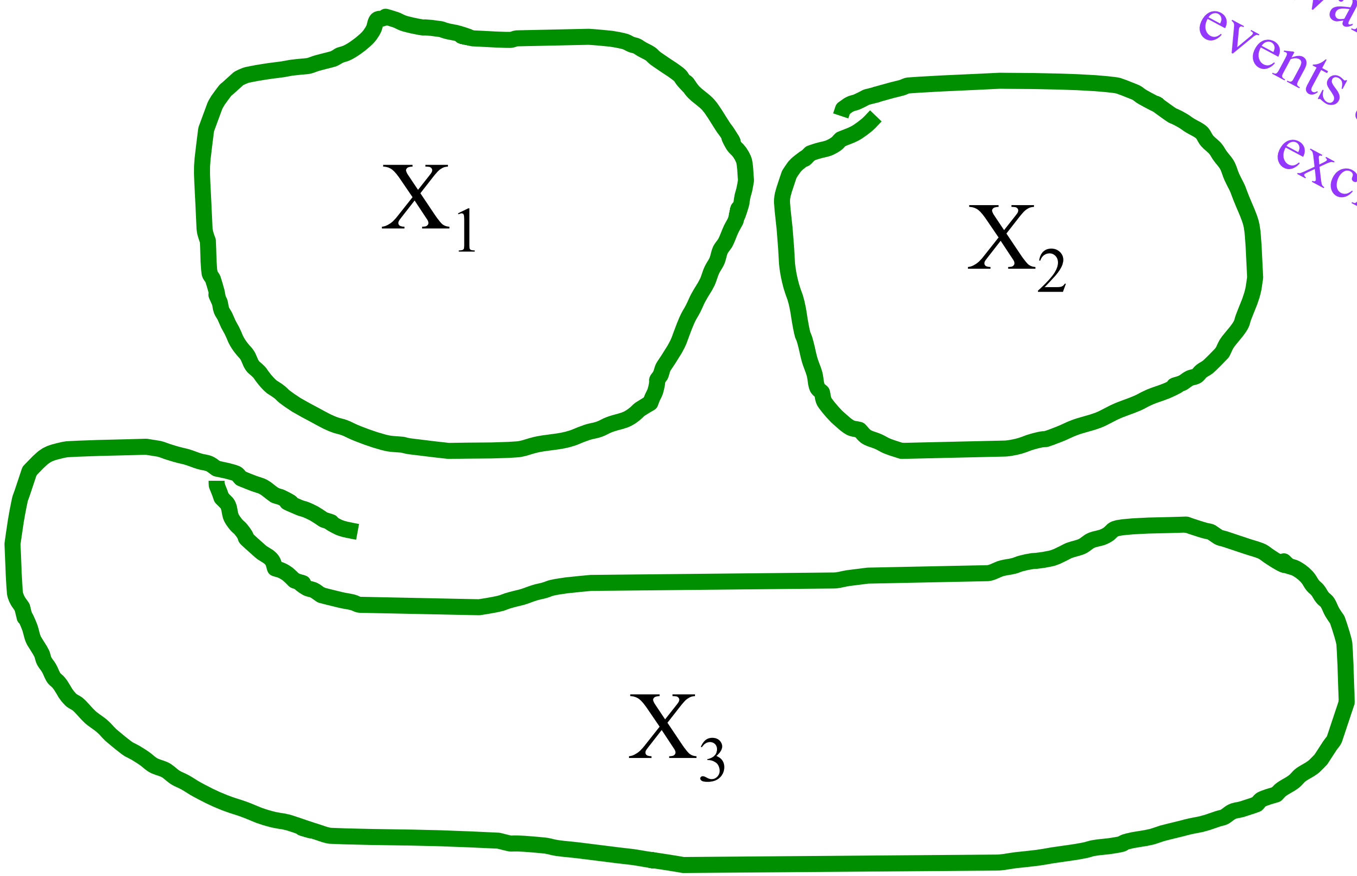
OR with Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

OR with Many Mutually Exclusive Events



Wahoo! All my events are mutually exclusive

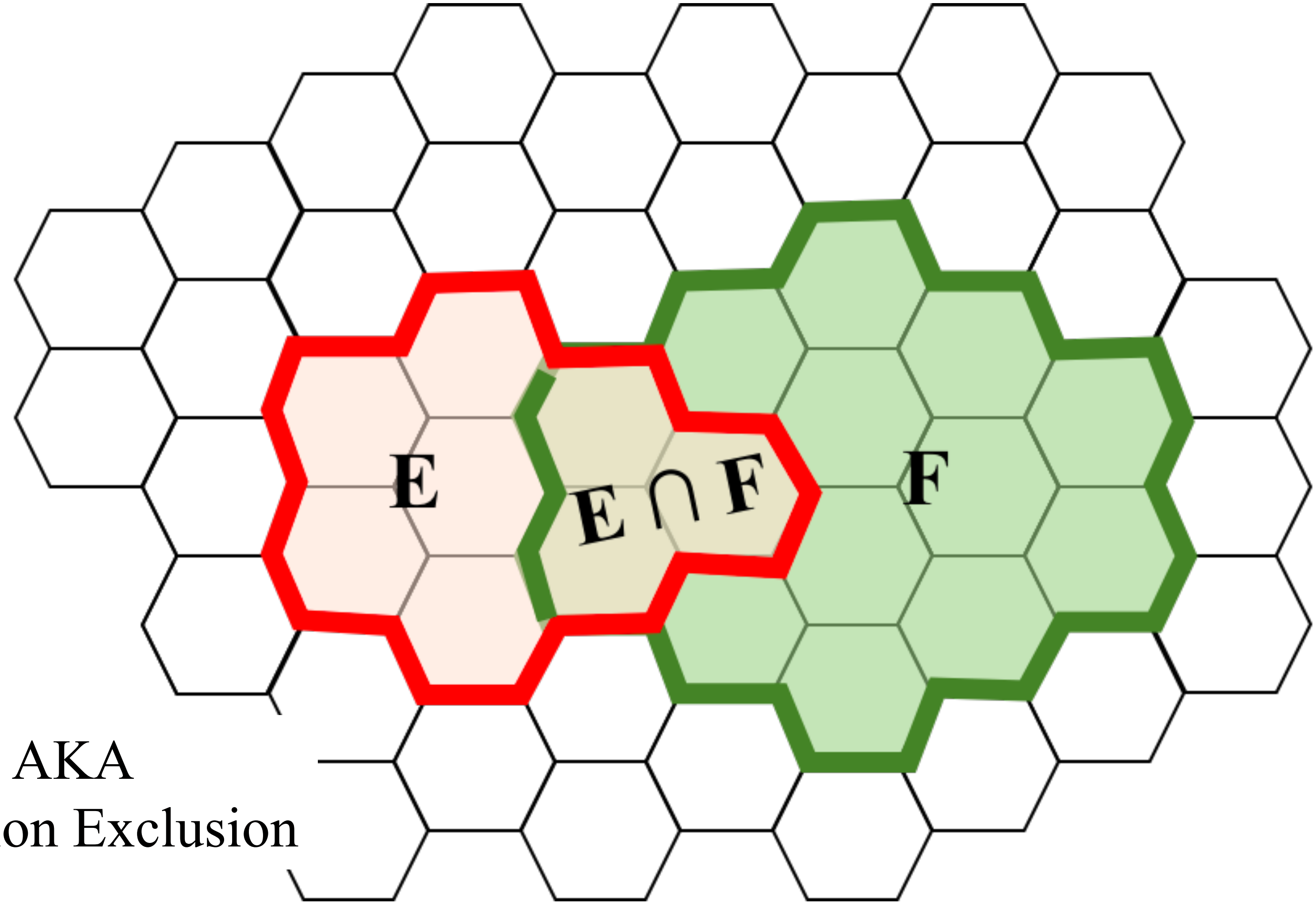
$$P(X_1 \cup X_2 \cup \cdots \cup X_n) = \sum_{i=1}^{n} P(X_i)$$

If events are *mutually exclusive* probability of OR is easy!

What about when they are not

Mutually exclusive?

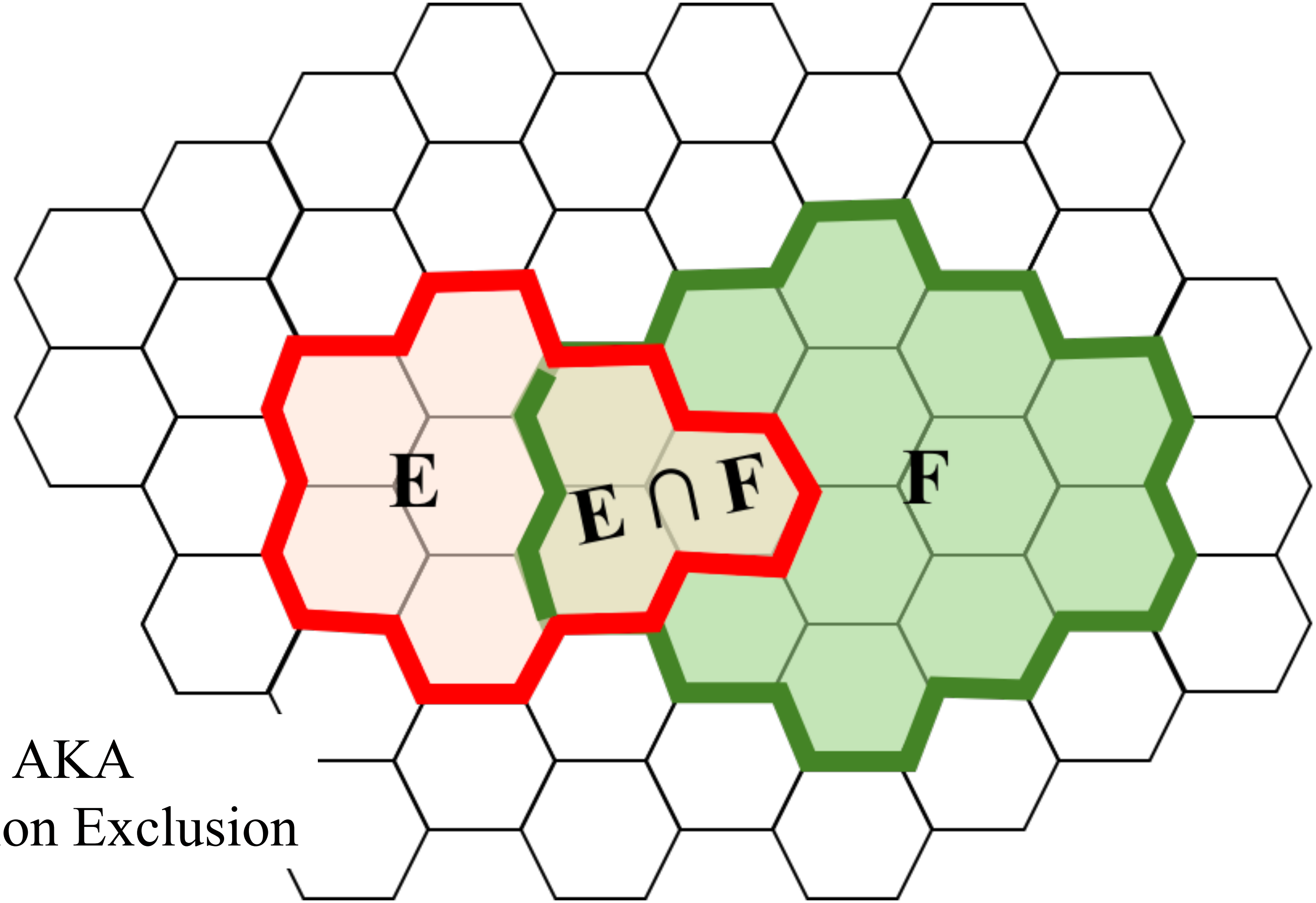
OR without Mutually Exclusivity



AKA
Inclusion Exclusion

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

OR without Mutually Exclusivity



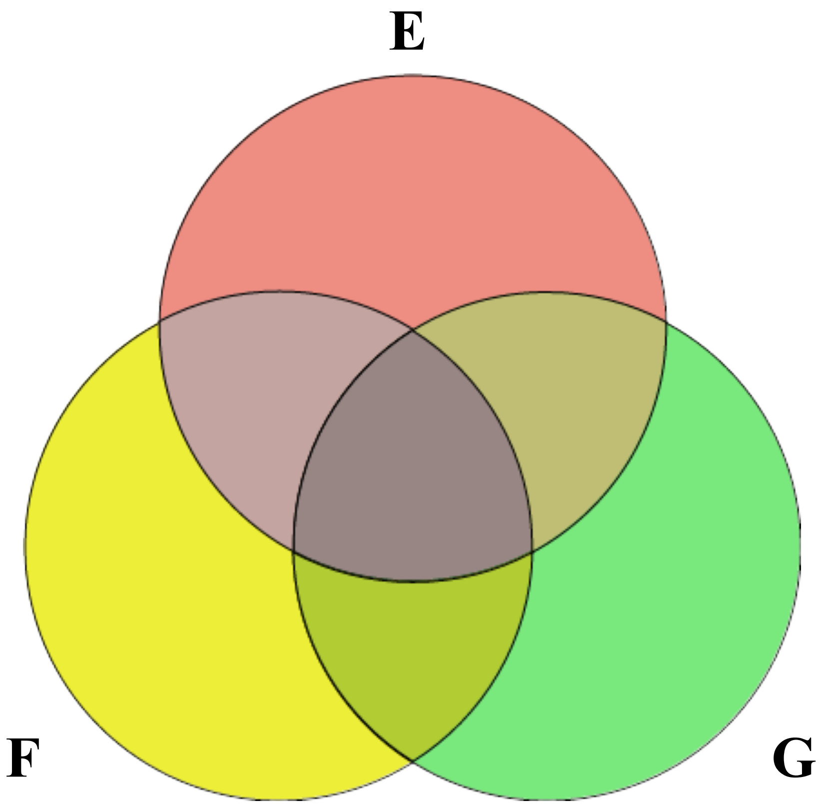
AKA
Inclusion Exclusion

$$P(E \cup F) = \frac{8}{50} + \frac{14}{50} - \frac{3}{50} = \frac{19}{50}$$

More than two sets?

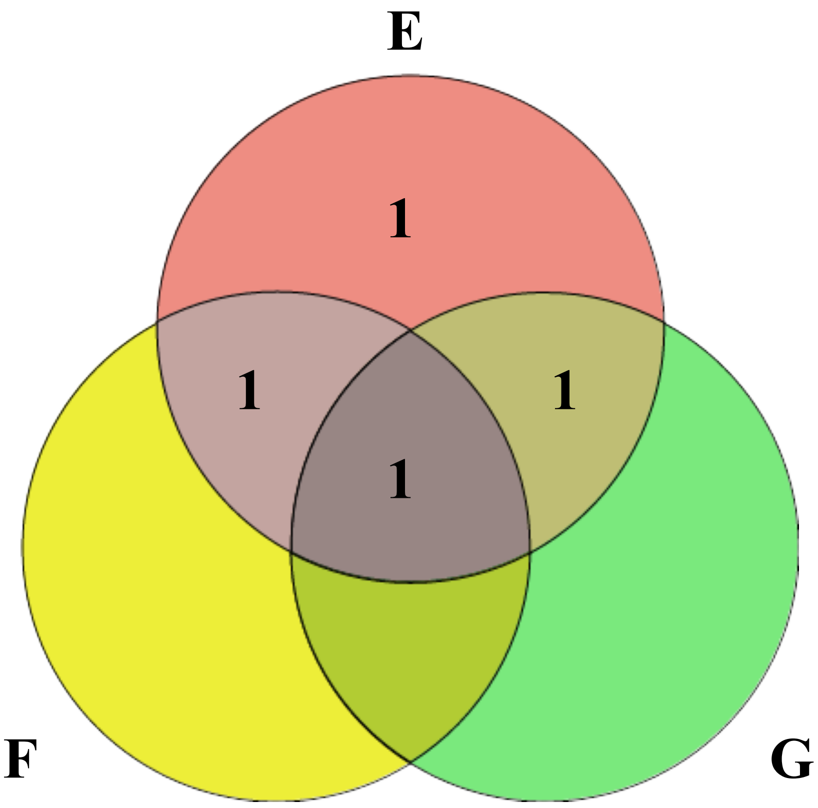
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) =$$



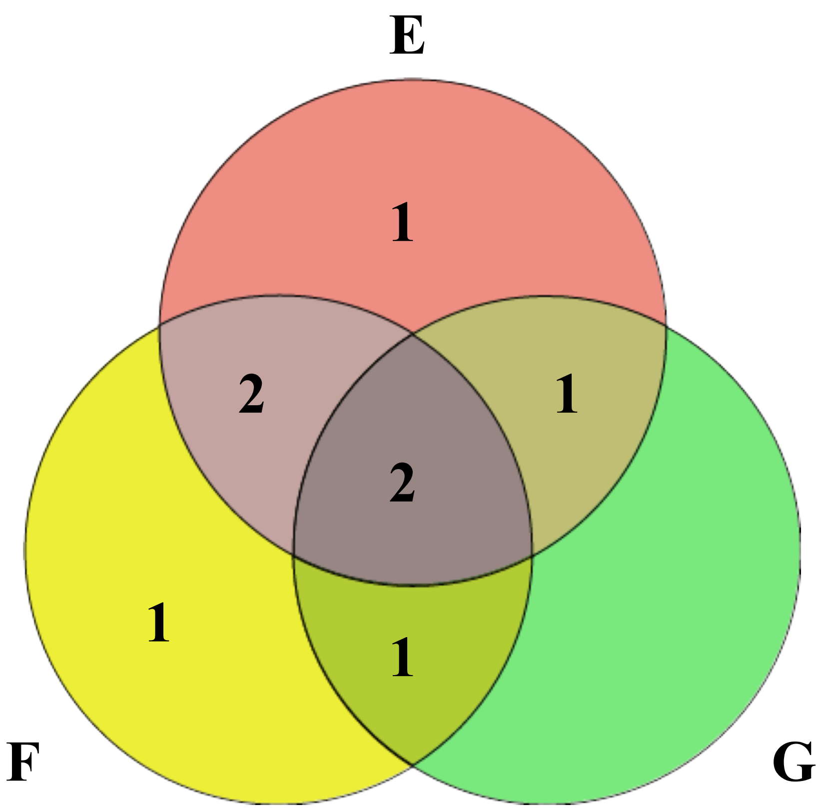
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E)$$



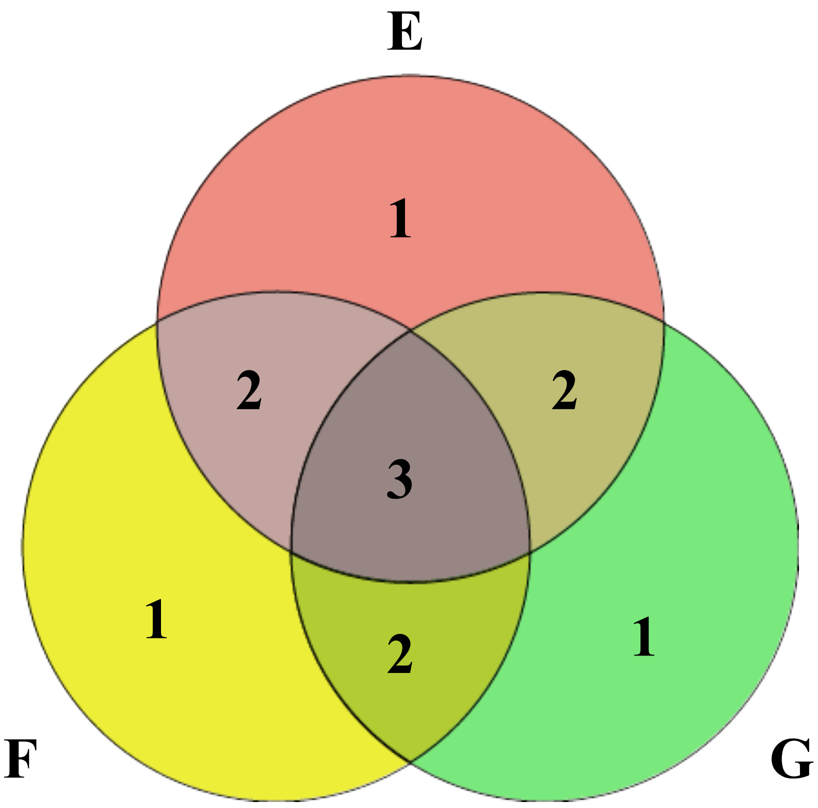
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E) + P(F)$$



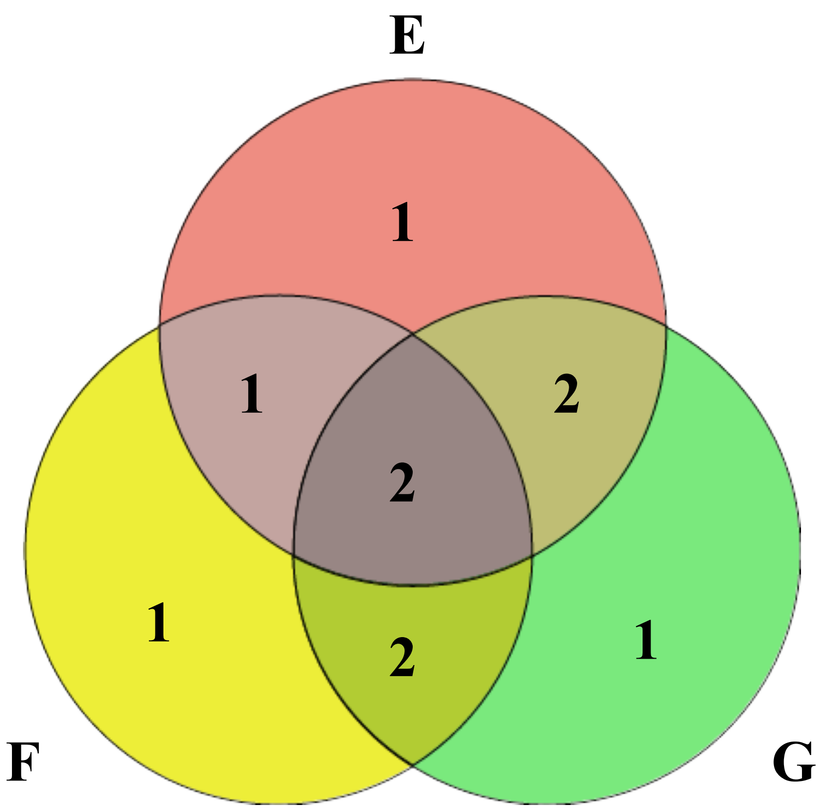
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$



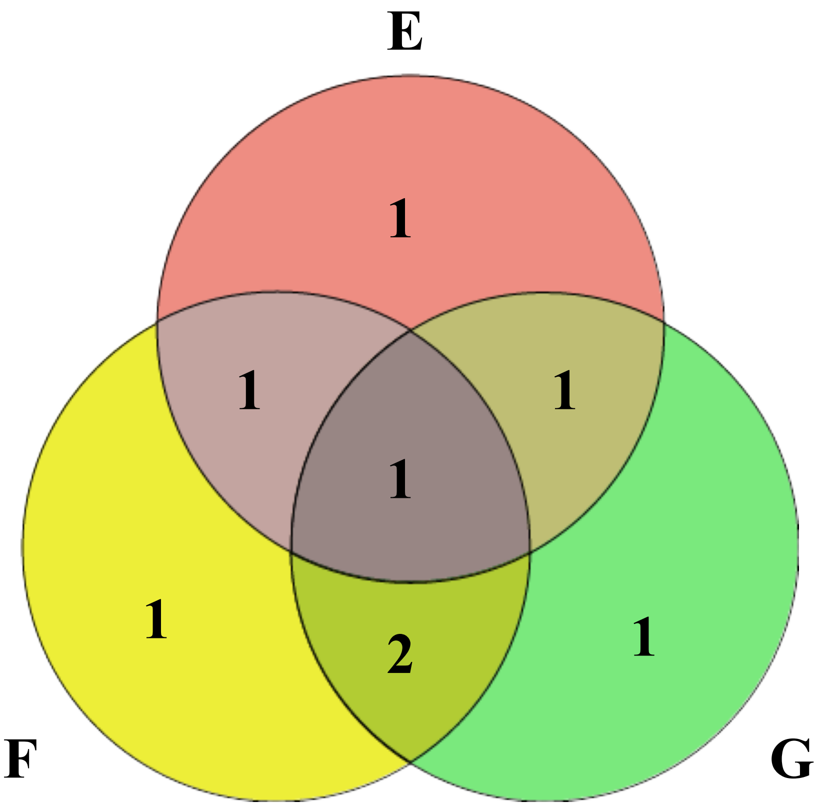
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF)$$



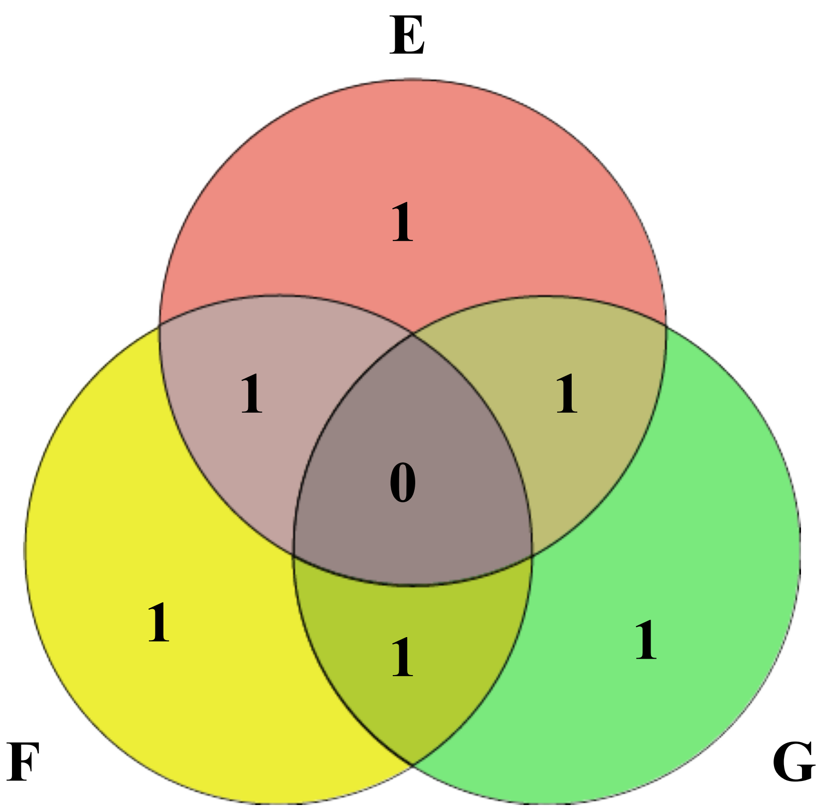
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG)$$



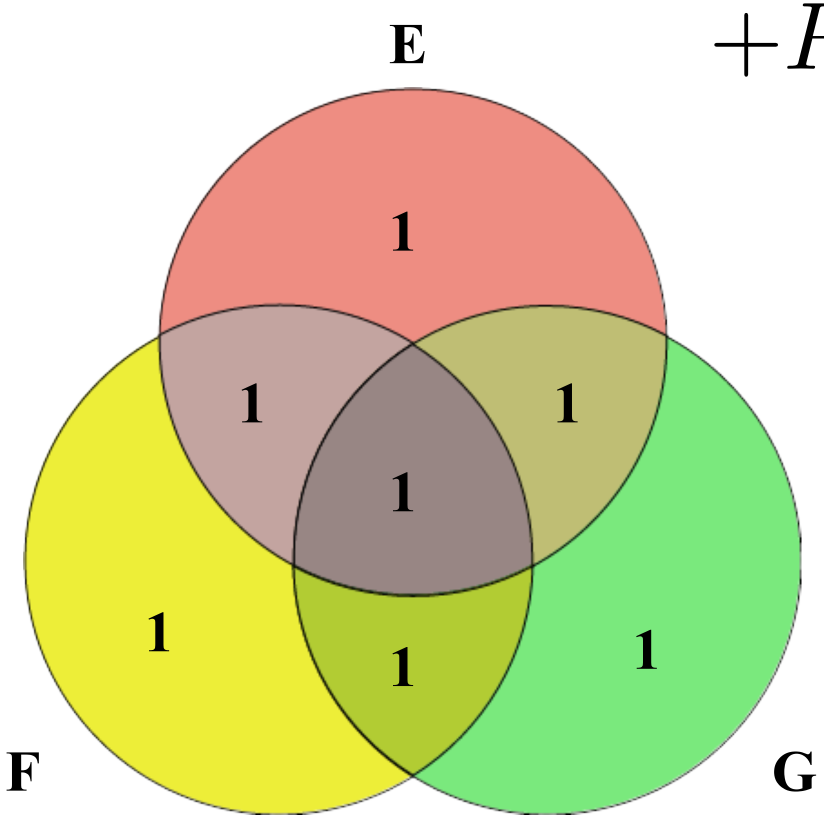
Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG) - P(FG)$$



Inclusion Exclusion with Three Sets

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG) - P(FG)$$
$$+P(EFG)$$



General Inclusion Exclusion

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} Y_r$$

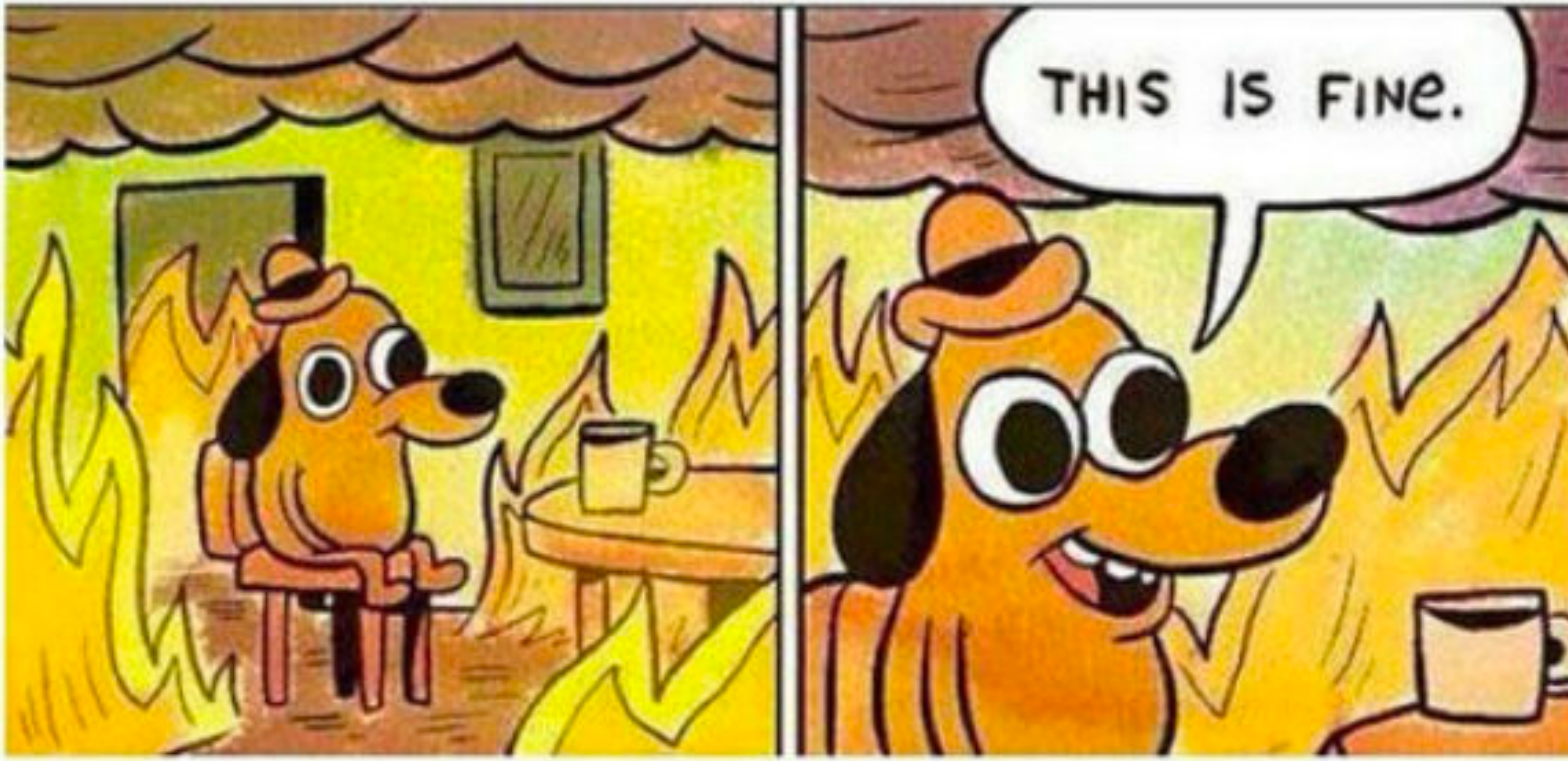
* Where Y_r is the sum, for all combinations of r events, of the probability of the union those events.

Y_1 = Sum of all events on their own $\sum_{i} P(E_i)$

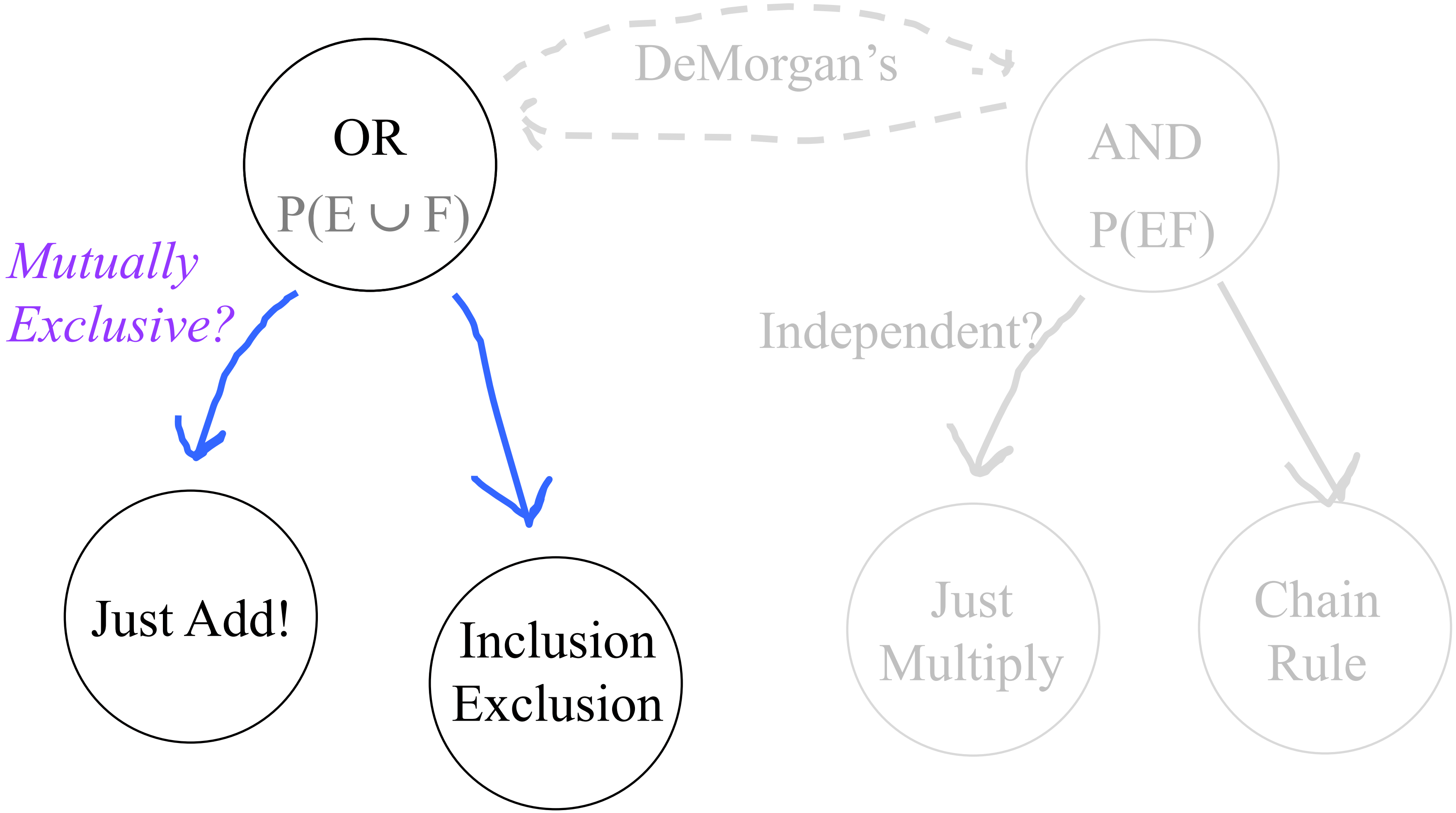
Y_2 = Sum of all pairs of events $\sum_{i,j} P(E_i \cap E_j)$ s.t. $i \neq j$

Y_3 = Sum of all triples of events $\sum_{i,j,k} P(E_i \cap E_j \cap E_k)$ s.t. $i \neq j, j \neq k, i \neq k$

General Inclusion Exclusion

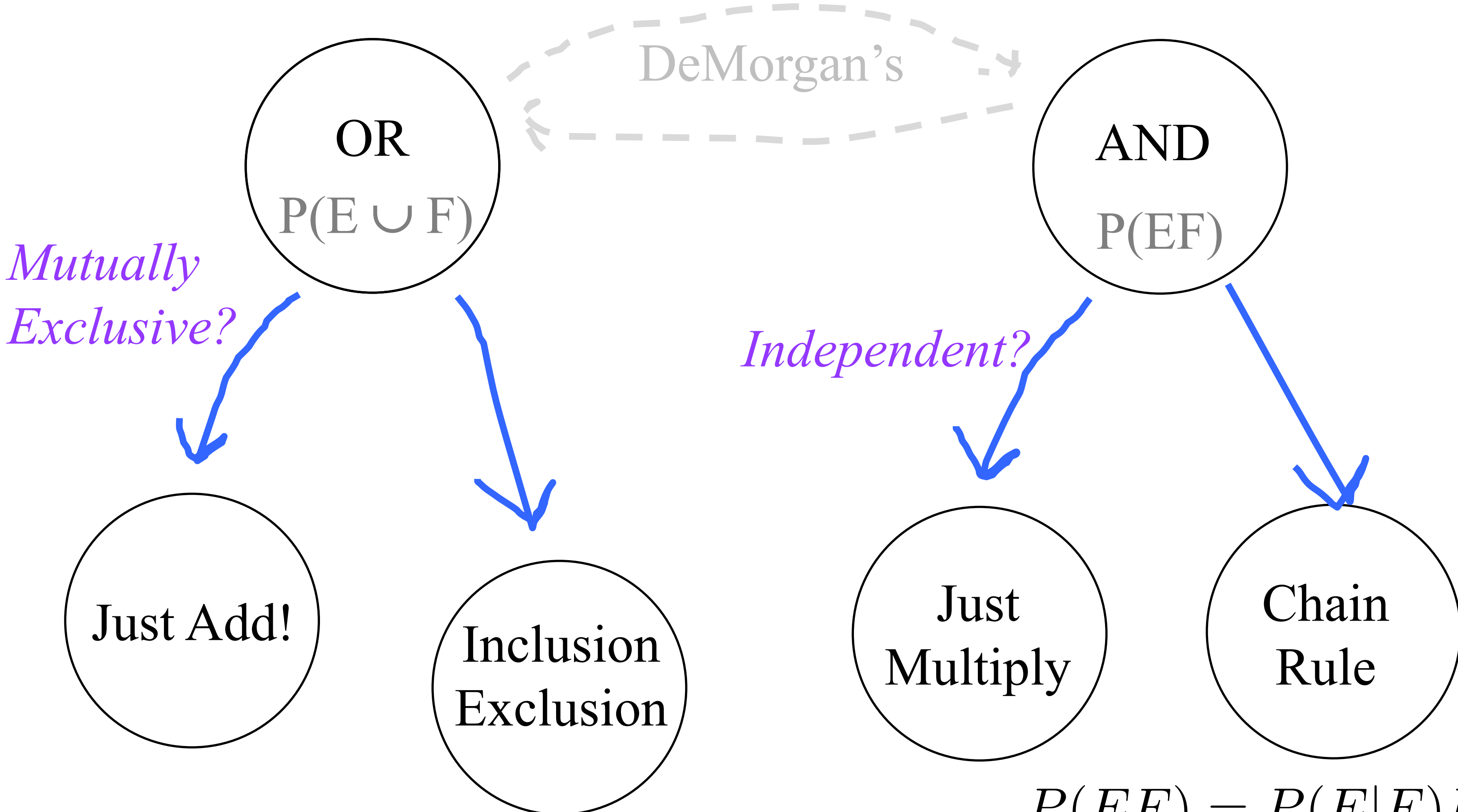


Today



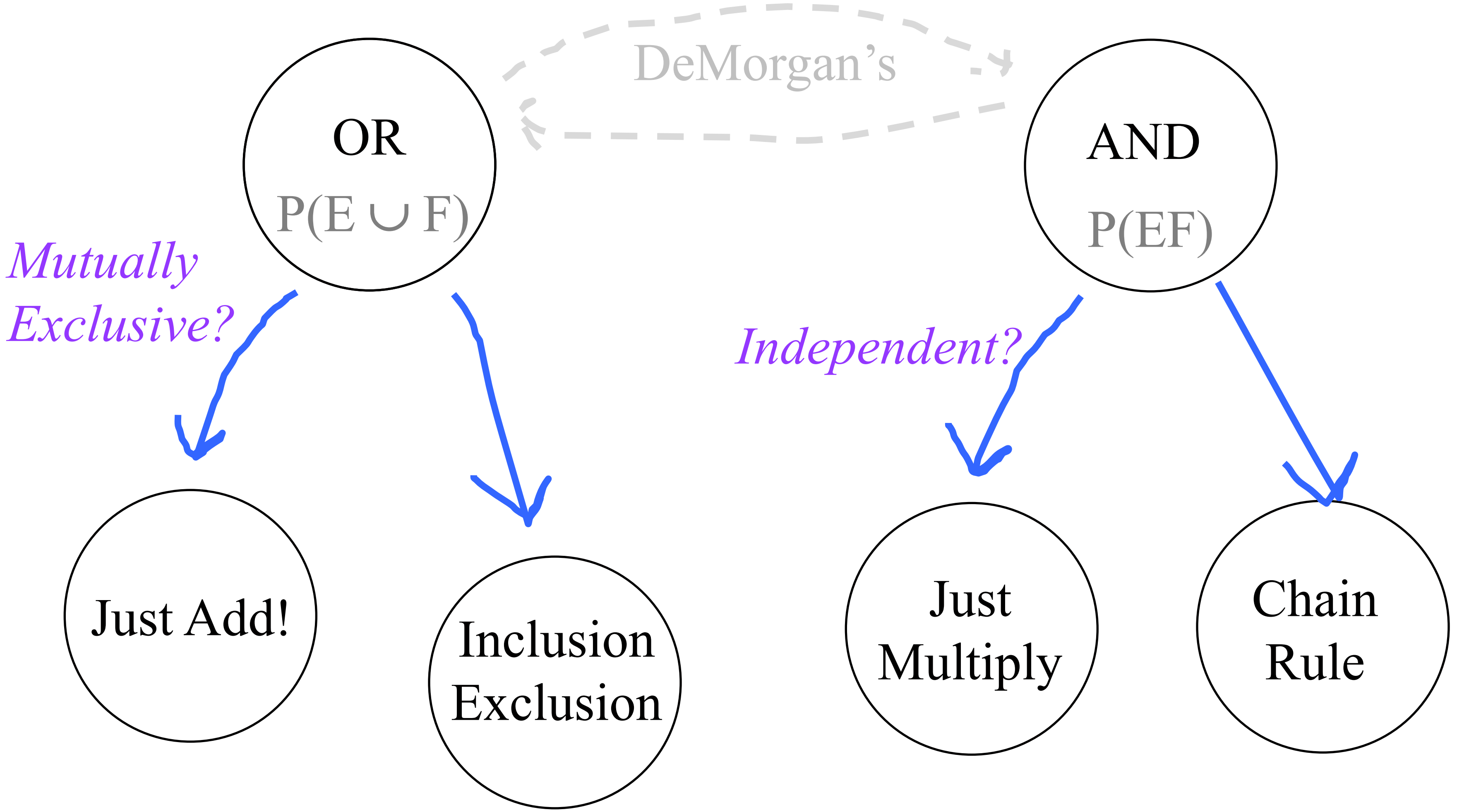
DeMorgan's
OR
P(E ∪ F)
AND
P(EF)
Mutually
Exclusive?
Independent?
Just Add!
Inclusion
Exclusion
Just
Multiply
Chain
Rule

Today



$$P(EF) = P(E|F)P(F)$$

Today



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Just Multiply
Chain Rule

Probability of “AND”


We the People
insure domestic Tranquility, provide for the common
and our Posterity, do ordain and establish this
Section. 1. All legislative Powers

Independence

Two events A and B are called **independent** if:

$$P(AB) = P(A)P(B)$$

Otherwise, they are called **dependent** events

If events are *independent* probability of AND is easy!

*You will need to use this “trick” with high probability

Intuition through proofs

Let A and B be independent

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Definition of conditional probability

$$= \frac{P(A)P(B)}{P(B)}$$

Since A and B are independent

$$= P(A)$$

Taking the bus to cancel city

Knowing that event B happened, doesn't change our belief that A will happen.

Dice, Our Misunderstood Friends

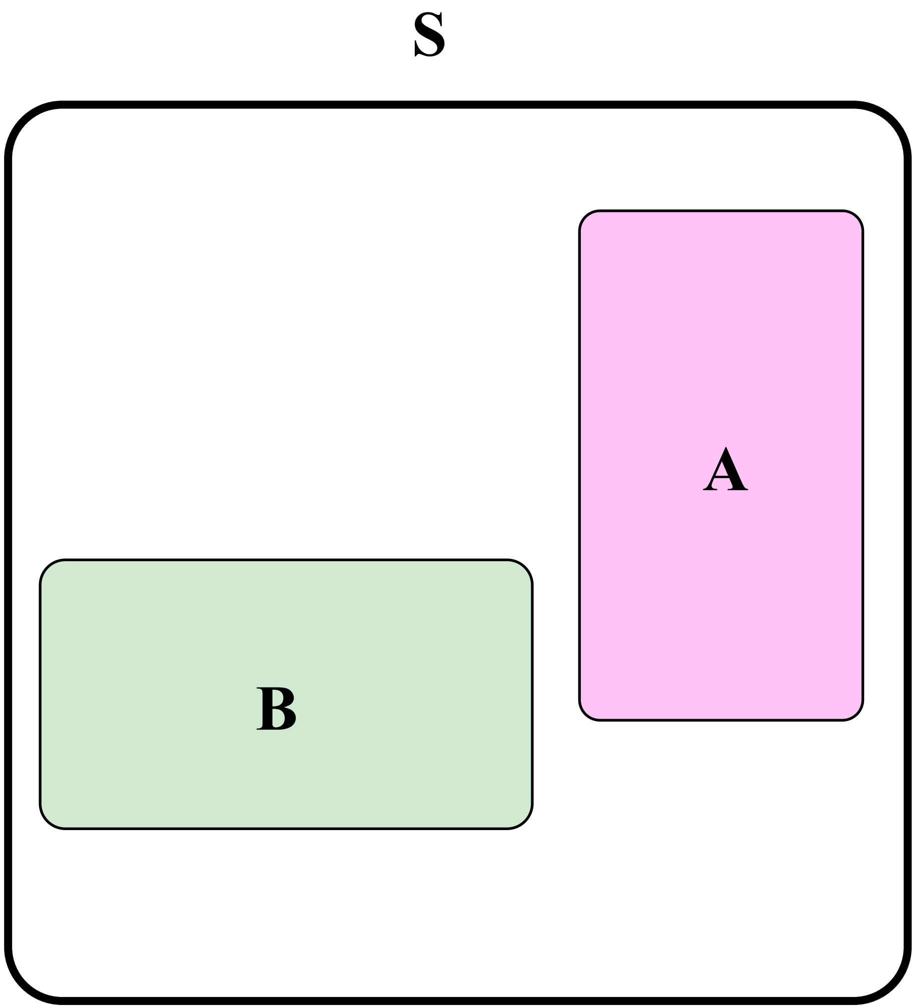
- Roll two 6-sided dice, yielding values D_1 and D_2
  - Let E be event: $D_1 = 1$
  - Let F be event: $D_2 = 1$

- What is P(E), P(F), and P(EF)?
  - $P(E) = 1/6$, $P(F) = 1/6$, $P(EF) = 1/36$
  - $P(EF) = P(E)\,P(F)$ $\rightarrow$ E and F <u>*independent*</u>

- Let G be event: $D_1 + D_2 = 5$ $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
- What is P(E), P(G), and P(EG)?
  - $P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(EG) = 1/36$
  - $P(EG) \neq P(E)\,P(G)$ $\rightarrow$ E and G <u>*dependent*</u>

What does independence look like?

Independence?



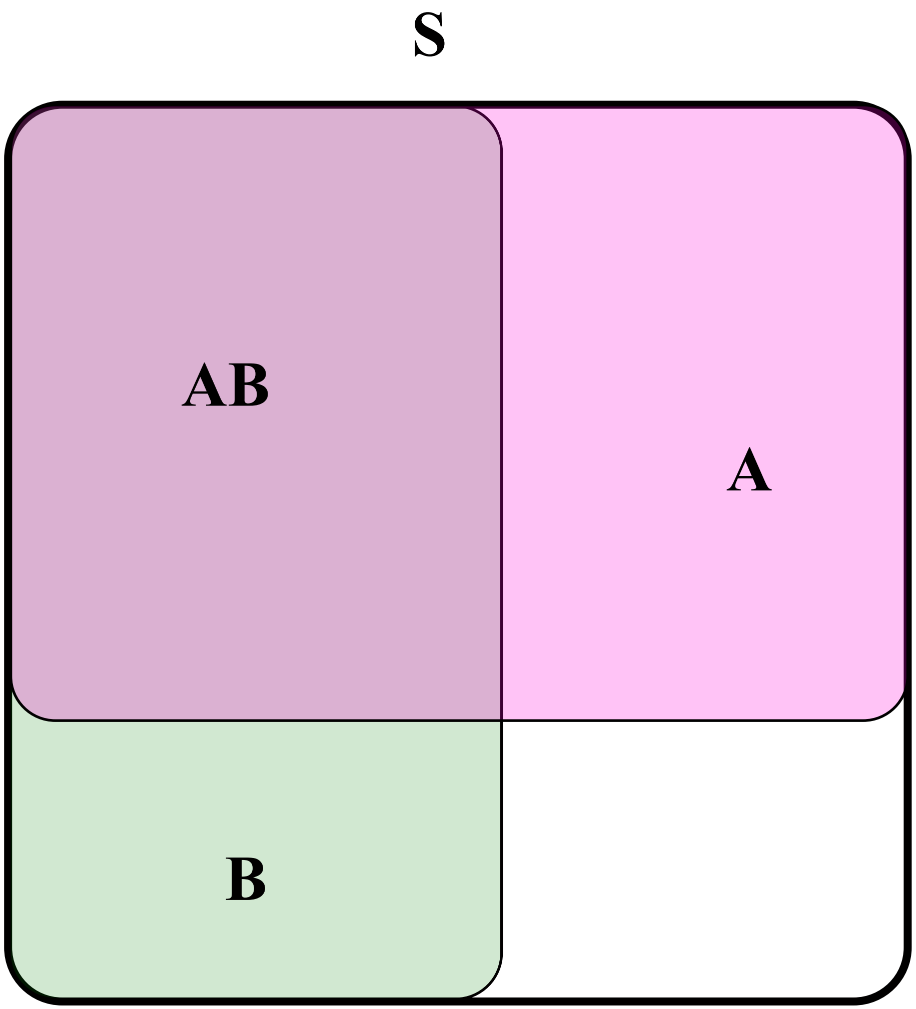
Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

0

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

More Intuition through proofs:

Independence

Given independent events A and B, prove that A and B^C are independent

We want to show that $P(AB^C) = P(A)P(B^C)$

$$\begin{aligned} P(AB^C) &= P(A) - P(AB) && \text{By Intersection Rule} \\ &= P(A) - P(A)P(B) && \text{By independence} \\ &= P(A)[1 - P(B)] && \text{Factoring} \\ &= P(A)P(B^C) && \text{Since } P(B) + P(B^C) = 1 \end{aligned}$$

So if A and B are independent A and B^C are also independent

Generalization

Generalized Independence

- General definition of Independence:

  Events E_1 , E_2 , ..., E_n are independent if for every subset with r elements (where $r \leq n$) it holds that:

$$P(E_{1'}E_{2'}E_{3'}\ldots E_{r'}) = P(E_{1'})P(E_{2'})P(E_{3'})\ldots P(E_{r'})$$

- <u>Example</u>: outcomes of *n* separate flips of a coin are all independent of one another
  - Each flip in this case is called a “trial” of the experiment

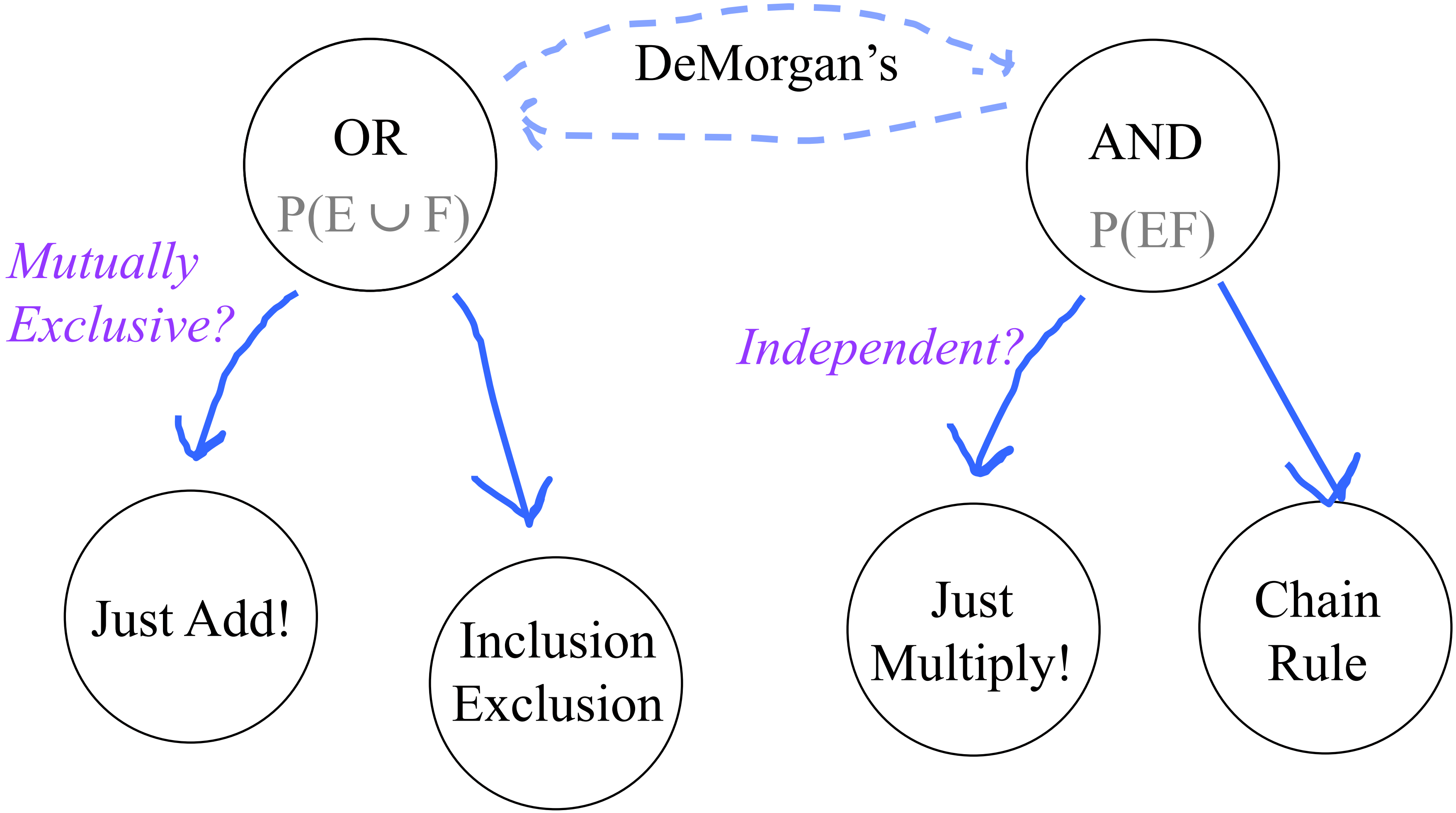
Math > Intuition

Two Dice

- Roll two 6-sided dice, yielding values D_1 and D_2
  - Let E be event: $D_1 = 1$
  - Let F be event: $D_2 = 6$
  - Are E and F independent? Yes!
- Let G be event: $D_1 + D_2 = 7$
  - Are E and G independent? Yes!
  - $P(E) = 1/6$, $P(G) = 1/6$, $P(E\,G) = 1/36$ [roll (1, 6)]
  - Are F and G independent? Yes!
  - $P(F) = 1/6$, $P(G) = 1/6$, $P(F\,G) = 1/36$ [roll (1, 6)]
  - Are E, F and G independent? No!
  - $P(EFG) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

New Ability

Today



DeMorgan's
OR
P(E ∪ F)
AND
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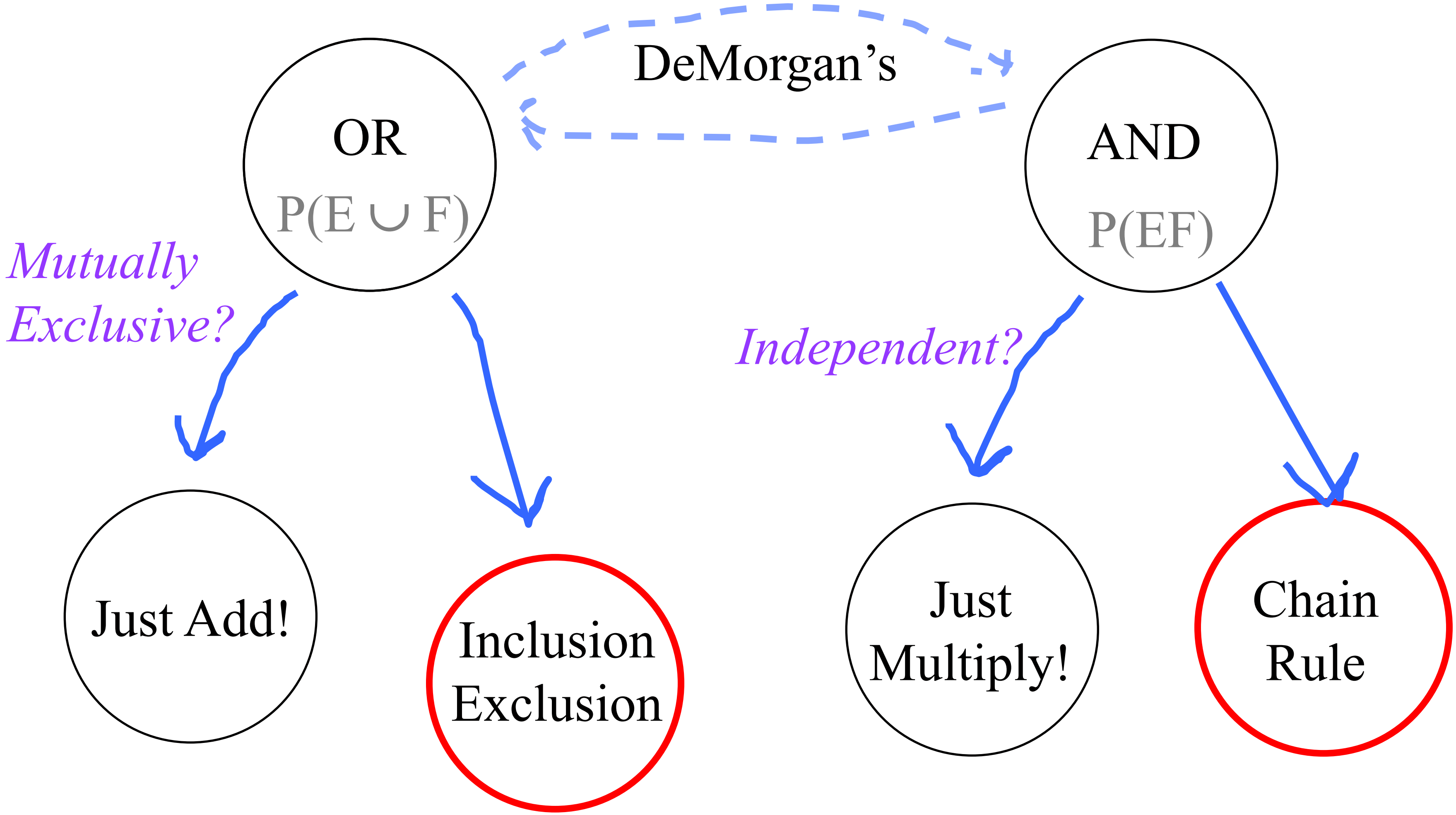
Mutually Exclusive?
Independent?
Just Add!
Inclusion Exclusion
Just Multiply!
Chain Rule

Today


DeMorgan's
OR
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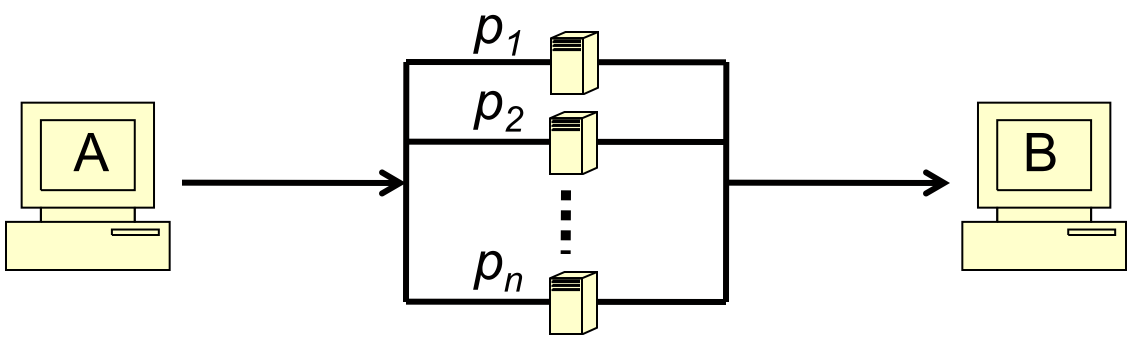
Chain Rule

Use the two properties (mutual exclusion and independence)

Sending a Message Through Network

- Consider the following parallel network:



  - n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
  - E = functional path from A to B exists. What is P(E)?

- Solution:
  - P(E) $= 1 - $ P(all routers fail)

$$= 1 - (1 - p_1)(1 - p_2)\ldots(1 - p_n)$$

$$= 1 - \prod_{i=1}^{n}(1 - p_i)$$

Coin Flips

- Say a coin comes up heads with probability p
  - Each coin flip is an independent trial

- P(*n* heads on *n* coin flips) = p^n
- P(*n* tails on *n* coin flips) = $(1 - p)^n$

- P(first *k* heads, then *n* – *k* tails) = $p^k(1-p)^{n-k}$

- P(exactly *k* heads on *n* coin flips) =?

Explain…

P(exactly k heads on n coin flips)? $\binom{n}{k} p^k (1-p)^{n-k}$

Think of the flips as ordered:

Ordering 1: T, H, H, T, T, T….

Ordering 2: H, T, H, T, T, T….

And so on…

The coin flips are independent!

$P(F_i) = p^k (1-p)^{n-k}$

Let's make each ordering with k heads an event… F_i

P(exactly k heads on n coin flips) = P(any one of the events)

P(exactly k heads on n coin flips) = P(F_1 or F_2 or F_3…)

Those events are mutually exclusive!

Hash Tables

- *m* strings are hashed (unequally) into a hash table with *n* buckets
  - Each string hashed is an independent trial, with probability p_i of getting hashed to bucket *i*
  - E = at least one string hashed to first bucket
  - What is P(E)?
- Solution

To the white board!

Yet More Hash Tables

- m strings are hashed (unequally) into a hash table with n buckets
  - Each string hashed is an independent trial, with probability p_i of getting hashed to bucket i
  - E = At least 1 of buckets 1 to k has ≥ 1 string hashed to it
- Solution
  - F_i = at least one string hashed into i-th bucket
  - $P(E) \quad = P(F_1 \cup F_2 \cup \ldots \cup F_k) = 1 - P((F_1 \cup F_2 \cup \ldots \cup F_k)^c)$

    $= 1 - P(F_1^c F_2^c \ldots F_k^c)$ (DeMorgan's Law)
  - $P(F_1^c F_2^c \ldots F_k^c) = P(\text{no strings hashed to buckets 1 to } k)$

    $= (1 - p_1 - p_2 - \ldots - p_k)^m$
  - $P(E) \quad = 1 - (1 - p_1 - p_2 - \ldots - p_k)^m$

No, Really, More Hash Tables

- *m* strings are hashed (unequally) into a hash table with *n* buckets
  - Each string hashed is an independent trial, with probability p_i of getting hashed to bucket *i*
  - E = Each of buckets 1 to *k* has ≥ 1 string hashed to it
- Solution
  - F_i = at least one string hashed into *i*-th bucket
  - $P(E) \quad = P(F_1 F_2 \ldots F_k) = 1 - P((F_1 F_2 \ldots F_k)^c)$

$$= 1 - P(F_1^c \cup F_2^c \cup \ldots \cup F_k^c) \qquad \text{(DeMorgan's Law)}$$

$$= 1 - P\left(\bigcup_{i=1}^{k} F_i^c\right) = 1 - \sum_{r=1}^{k} (-1)^{(r+1)} \sum_{i_1 < \ldots < i_r} P(F_{i_1}{}^c F_{i_2}{}^c \ldots F_{i_r}{}^c)$$

where $P(F_{i_1}{}^c F_{i_2}{}^c \ldots F_{i_r}{}^c) = (1 - p_{i_1} - p_{i_2} - \ldots - p_{i_r})^m$

Phew!

Conditional Paradigm

- Recall:

$$P(A\ B) = P(B\ A\ )$$

$$P(A\ B) = P(A\ |\ B\ )\ P(B)$$

Conditional Paradigm

- For any events A, B, and E, you can condition <u>consistently</u> on E, and these formulas still hold:

$$P(A\,B \mid E) = P(B\,A \mid E)$$

$$P(A\,B \mid E) = P(A \mid B\,E)\,P(B \mid E)$$

- Can think of E as "everything you already know"
- Formally, $P(\bullet \mid E)$ satisfies 3 axioms of probability

BAE's Theorem?

$$P(A \mid B\,E) = \frac{P(B \mid A\,E)\,P(A \mid E)}{P(B \mid E)}$$

Conditional Independence

- Two events E and F are called **conditionally independent given G**, if

  $$P(E\,F \mid G) = P(E \mid G)\,P(F \mid G)$$

  Or, equivalently: $P(E \mid F\,G) = P(E \mid G)$

NETFLIX

And Learn

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$P(E)$

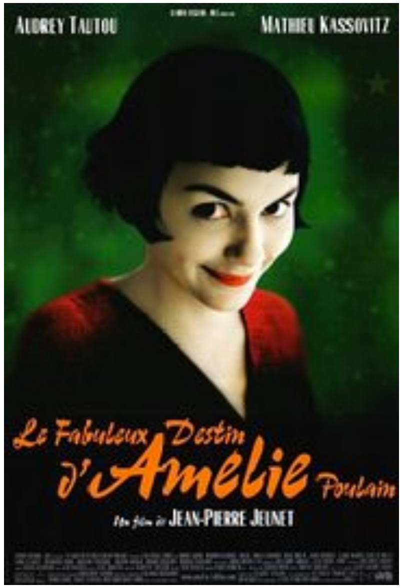
$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

$P(E) = 10{,}234{,}231 \,/\, 50{,}923{,}123 = 0.20$

Netflix and Learn

What is the probability that a user will watch Life is Beautiful, given they watched Amelie?

$P(E|F)$



$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\#\text{people who watched both}}{\#\text{people who watched } F}$$

$P(E|F) = 0.42$

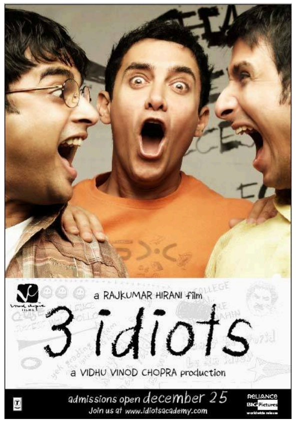
Conditioned on liking a set of movies?

Netflix and Learn

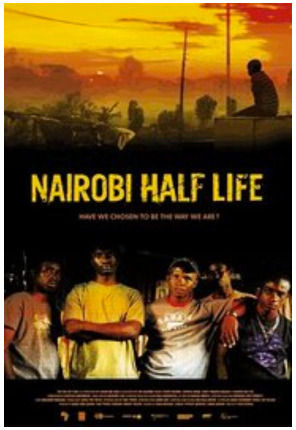
Each event corresponds to liking a particular movie

E_1

E_2



E_3



E_4

$$P(E_4 | E_1, E_2, E_3)?$$

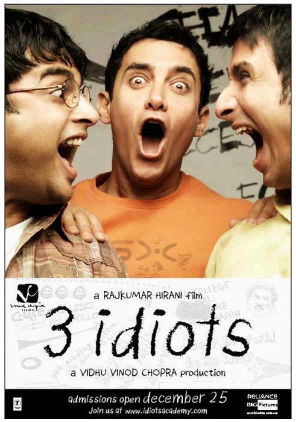
Is E_4 independent of E_1, E_2, E_3?

Netflix and Learn

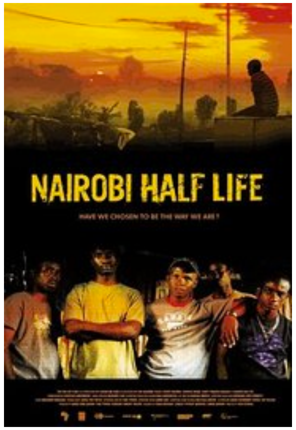
Is E_4 independent of E_1,E_2,E_3?

E_1

E_2



E_3



E_4

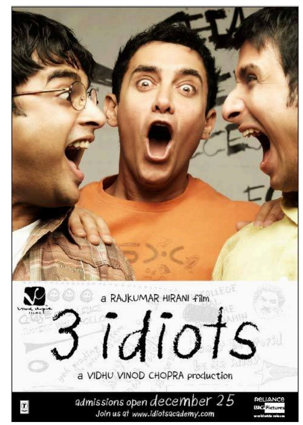
$$P(E_4|E_1, E_2, E_3) \stackrel{?}{=} P(E_4)$$

Netflix and Learn

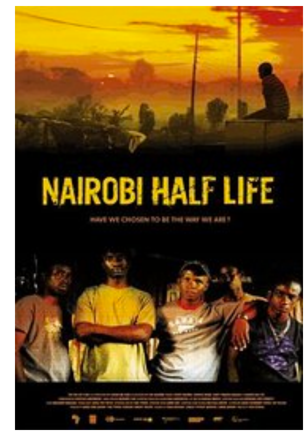
Is E_4 independent of E_1, E_2, E_3?

E_1

E_2



E_3



E_4

$$P(E_4 | E_1, E_2, E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)}$$

Netflix and Learn

- What is the probability that a user watched four particular movies?
  - There are 13,000 titles on Netflix
  - The user watches 30 random titles
  - E = movies watched include the given four.
- Solution:

Watch those four

Choose 24 movies not in the set

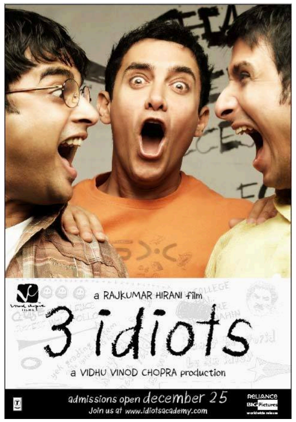
$$P(E) = \frac{\binom{4}{4}\binom{12996}{24}}{\binom{13000}{30}} = 10^{-11}$$

Choose 30 movies from netflix

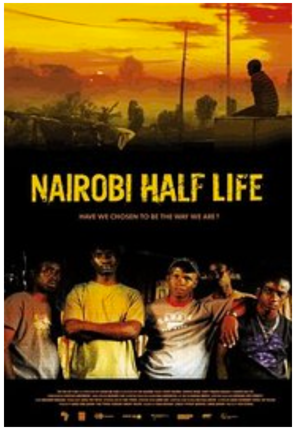
Netflix and Learn

E_1

E_2



E_3

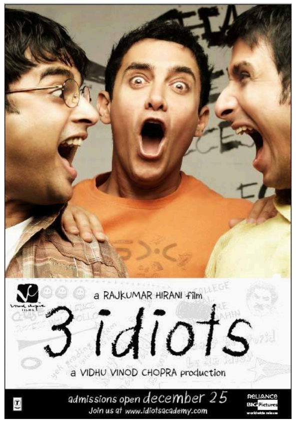


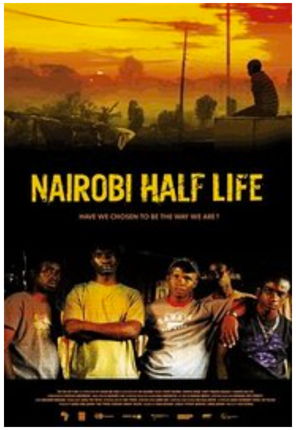
E_4

Netflix and Learn

K_1

Like foreign emotional comedies





E_1 E_2 E_3 E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix and Learn

K_1

Like foreign emotional comedies

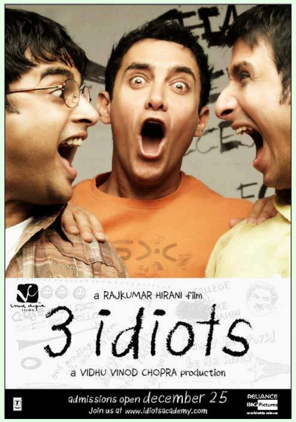
E_1 E_2 E_3 E_4

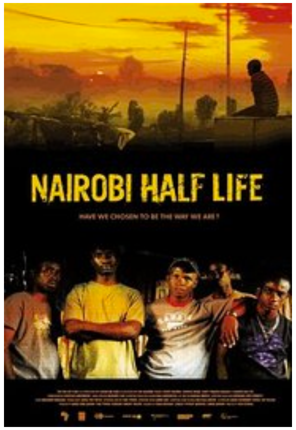
Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix and Learn

K_1

Like foreign emotional comedies





E_1 ***E_2*** ***E_3*** *E_4*

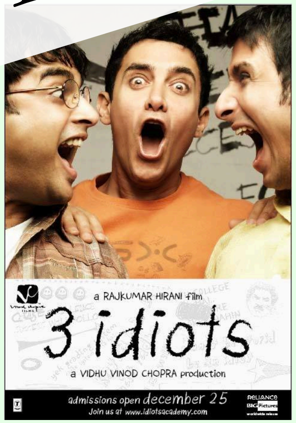
Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

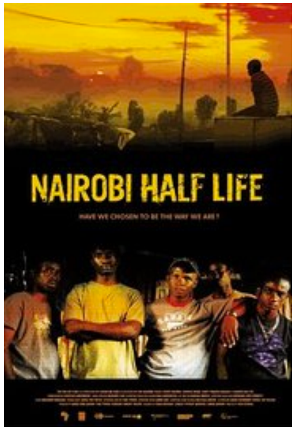
Netflix and Learn

K₁

Like foreign emotional comedies

$$P(E_4|E_1, E_2, E_3, K_1) = P(E_4|K_1)$$





E_1 E_2 E_3 E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Conditional independence is a practical, real world way of decomposing hard probability questions.

Big Deal

"Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory"

-Judea Pearl wins 2011 Turing Award, *"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"*


G_1
G_2
G_3
G_4
G_5
T

Discovered Pattern